

The Kinematic Equations

Given that: $\vec{a} = \frac{d\vec{v}}{dt}$ equation (1)

$\vec{v} = \frac{d\vec{x}}{dt}$ equation (2)

Integrate equation (1):

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \int_0^t \vec{a} dt = \int_0^t d\vec{v}$$

assume that $\vec{a} = \text{constant}$, so

$$\begin{aligned} \int_0^t \vec{a} dt &= \int_0^t d\vec{v} \Rightarrow \vec{a} \int_0^t dt = \int_0^t d\vec{v} \\ \vec{a} [t]_0 &= [\vec{v}]_0 \\ \vec{a} [t - t_0] &= [\vec{v} - \vec{v}_0] \end{aligned}$$

assume that $t_0 = 0$ and rearrange leads to:

$$\boxed{\vec{v} = \vec{v}_0 + \vec{a}t}$$

kinematic equation (1)

Integrate equation (2):

$$\vec{v} = \frac{d\vec{x}}{dt} \Rightarrow \int_0^t \vec{v} dt = \int_0^t d\vec{x}$$

substitute kinematic equation (1) for $\vec{v}$:

$$\begin{aligned} \int_0^t \vec{v} dt &= \int_0^t d\vec{x} \Rightarrow \int_0^t (\vec{v}_0 + \vec{a}t) dt = \int_0^t d\vec{x} \\ [\vec{v}_0 t + \frac{1}{2} \vec{a} t^2]_0 &= [\vec{x}]_0 \\ [(\vec{v}_0 t + \frac{1}{2} \vec{a} t^2) - (\vec{v}_0 t_0 + \frac{1}{2} \vec{a} t_0^2)] &= [\vec{x} - \vec{x}_0] \end{aligned}$$

assume that $t_0 = 0$ and rearrange leads to:

$$\boxed{\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2}$$

kinematic equation (2)

Rearrange kinematic equation (1) to solve for t :

$$\vec{v} = \vec{v}_0 + \vec{a}t \Rightarrow t = \frac{\vec{v} - \vec{v}_0}{\vec{a}} \quad \text{equation (3)}$$

substitute equation (3) into kinematics equation (2):

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \Rightarrow \vec{x} = \vec{x}_0 + \vec{v}_0 \left(\frac{\vec{v} - \vec{v}_0}{\vec{a}} \right) + \frac{1}{2} \vec{a} \left(\frac{\vec{v} - \vec{v}_0}{\vec{a}} \right)^2$$

multiply by $\vec{a}$ and expand:

$$\vec{x} \vec{a} = \vec{x}_0 \vec{a} + \vec{v}_0 \vec{a} \left(\frac{\vec{v} - \vec{v}_0}{\vec{a}} \right) + \frac{1}{2} \vec{a} \vec{a} \left(\frac{\vec{v} \vec{v} - 2 \vec{v} \vec{v}_0 + \vec{v}_0 \vec{v}_0}{\vec{a} \vec{a}} \right)$$

$$\vec{x} \vec{a} = \vec{x}_0 \vec{a} + \cancel{\vec{v}_0 \vec{v}} - \vec{v}_0 \vec{v}_0 + \frac{1}{2} \vec{v} \vec{v} - \cancel{\vec{v} \vec{v}_0} + \frac{1}{2} \vec{v}_0 \vec{v}_0 = -\frac{1}{2} \vec{v}_0 \vec{v}_0$$

multiply by 2 and rearrange

$$\vec{v} \vec{v} = \vec{v}_0 \vec{v}_0 + 2 \vec{a} (\vec{x} - \vec{x}_0)$$

$$\boxed{\vec{v}^2 = \vec{v}_0^2 + 2 \vec{a} (\Delta \vec{x})}$$

kinematics equation (3)

Rearrange kinematic equation (1) to solve for $\vec{a}$:

$$\vec{v} = \vec{v}_0 + \vec{a}t \Rightarrow \vec{a} = \frac{\vec{v} - \vec{v}_0}{t} \quad \text{equation (4)}$$

substitute equation (4) into kinematic equation (2):

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \Rightarrow \vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} \left(\frac{\vec{v} - \vec{v}_0}{t} \right) t^2$$

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} \vec{v} t - \frac{1}{2} \vec{v}_0 t = \frac{1}{2} \vec{v}_0 t$$

$$\vec{x} = \vec{x}_0 + \frac{1}{2} \vec{v}_0 t + \frac{1}{2} \vec{v} t$$

simplify

$$\boxed{\vec{x} = \vec{x}_0 + \left(\frac{\vec{v}_0 + \vec{v}}{2} \right) t}$$

kinematic equation (4)