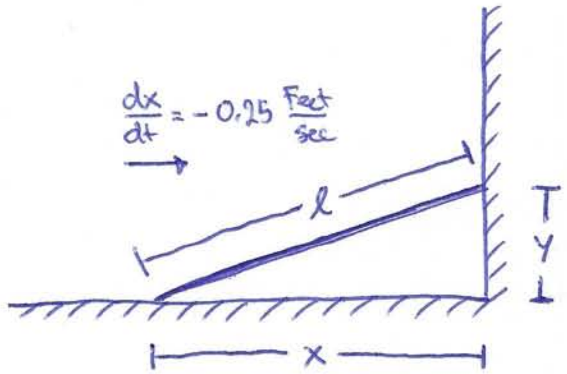
A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of 0.25 ft/sec. How fast is the top of the ladder moving up the wall when the ladder is 5 feet away from the wall?



$l = 15$ feet

Pythagoras:

$$l^2 = x^2 + y^2 \qquad (1)$$

$\frac{d}{dt}$

$$2l\cancel{\frac{dl}{dt}}^{\,0\ (l = \text{constant})} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad (2)$$

when ladder is 5 feet away from wall $x = 5$, use equ. (1)

$$l^2 = x^2 + y^2 \quad \Rightarrow \quad y = \sqrt{l^2 - x^2} = \sqrt{15^2 - 5^2} = 14.14 \text{ Feet}$$

solve for $\frac{dy}{dt}$ when $\frac{dx}{dt} = -0.25$ feet/sec, use equ. (2)

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{5}{14.14}(-0.25) = +0.088 \ \frac{\text{feet}}{\text{sec.}}$$

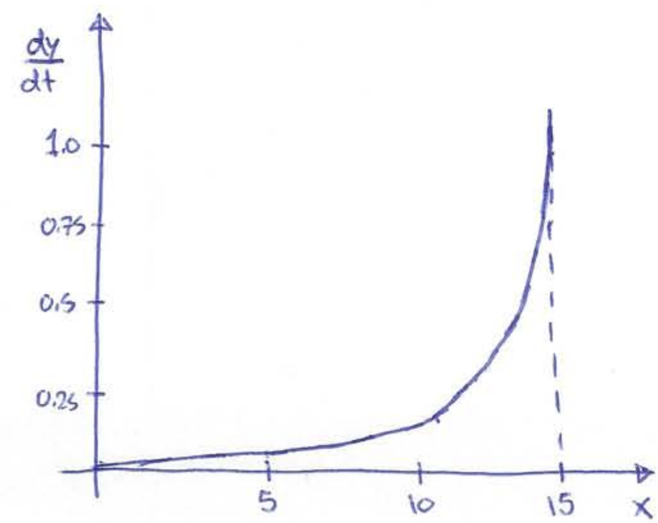
$$\boxed{\frac{dy}{dt} = 0.1 \ \frac{\text{foot}}{\text{sec}}}$$

sub (1) into (2):

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{\sqrt{l^2 + x^2}}\frac{dx}{dt}$$

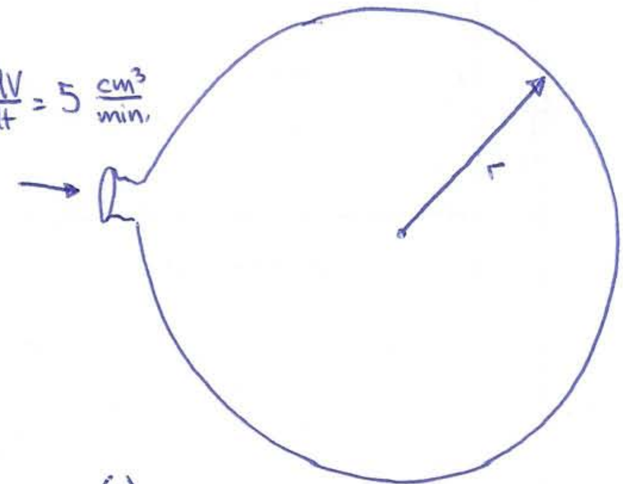
plot:

$$\frac{dy}{dx} = -\frac{x}{\sqrt{15^2 + x^2}}(-0.25) = \frac{0.25x}{\sqrt{15^2 + x^2}}$$



Air is being pumped into a spherical balloon at a rate of 5 cm³/min. Determine the rate at which the radius of the balloon is increasing when the radius of the balloon is 20 cm. The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$



Volume of sphere

$$V = \frac{4}{3}\pi r^3 \qquad (1)$$

$\frac{d}{dt}$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad (2)$$

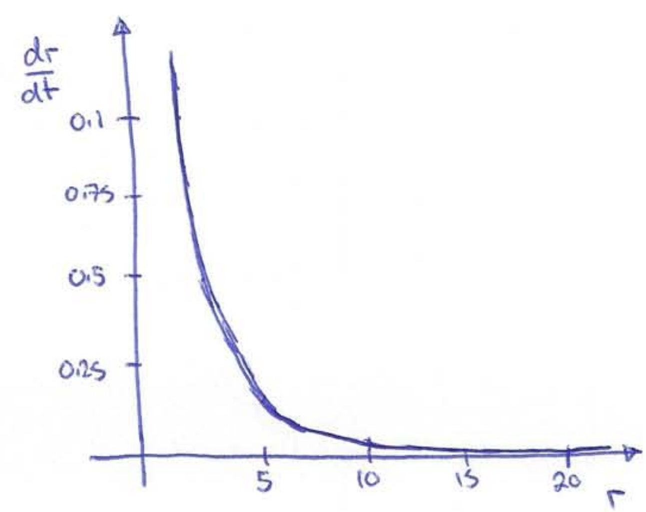
solve for $\frac{dr}{dt}$ when $r = 20\,cm$, use equ. (2)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi (20)^2}(5) = 0.000994 \frac{cm}{sec.}$$

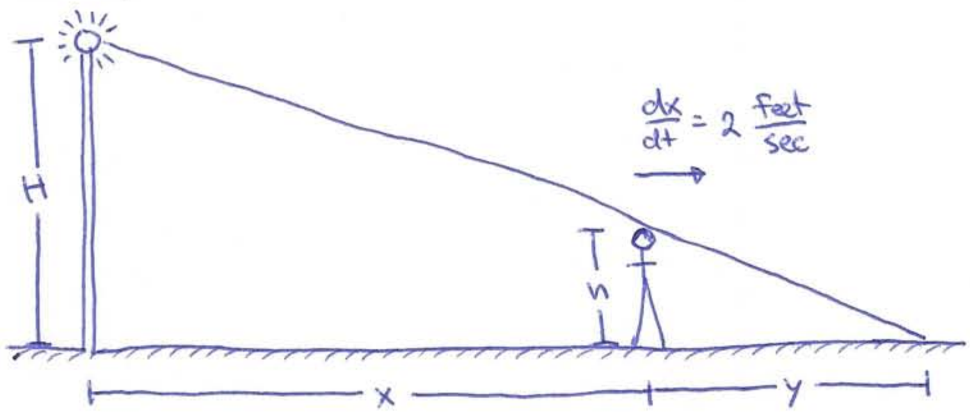
$$\boxed{\frac{dr}{dt} = 0.001 \frac{cm}{sec.}}$$

Plot:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}(5) = \frac{5}{4\pi r^2}$$



A light is on the top of a 12 foot tall pole and a 6 foot tall person is walking away from the pole at a rate of 2 ft/sec. (a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole? (b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?



Similar triangles

$$\frac{x+y}{H} = \frac{y}{h} \qquad (1)$$

$\frac{d}{dt}$

$$\frac{\frac{dx}{dt} + \frac{dy}{dt}}{H} = \frac{\frac{dy}{dt}}{h} \qquad (2)$$

Solve for $\frac{dy}{dt}$ when $x = 25$ feet, use equ. (2)

$$\frac{\frac{dx}{dt} + \frac{dy}{dt}}{H} = \frac{\frac{dy}{dt}}{h} \Rightarrow \frac{dy}{dt} = \frac{h}{H-h}\frac{dx}{dt} = \frac{6}{12-6}(2) = 2 \frac{\text{feet}}{\text{sec.}}$$

Relative velocities:

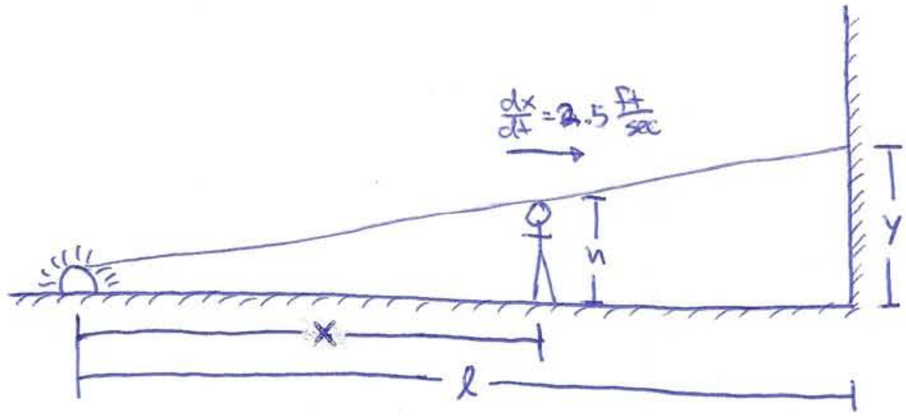
$\frac{dy}{dt}$ = velocity of shadow with respect to person

$\frac{dx}{dt}$ = velocity of person with respect to pole.

(a) shadow to pole $= \frac{dy}{dt} + \frac{dx}{dt} = 2 + 2 = 4 \frac{\text{feet}}{\text{sec}}$,

(b) shadow to person $= \frac{dy}{dt} = 2 \frac{\text{feet}}{\text{sec}}$

A spot light is on the ground 20 feet away from a wall and a 6 foot tall person is walking towards the wall at a rate of 2.5 ft/sec. How fast is the height of the shadow changing when the person is 8 feet from the wall?



Similar triangles

$$\frac{y}{\ell} = \frac{h}{x} \qquad (1)$$

$\frac{d}{dt}$

$$\frac{\frac{dy}{dt}}{\ell} = -\frac{h}{x^2}\frac{dx}{dt} \qquad (2)$$

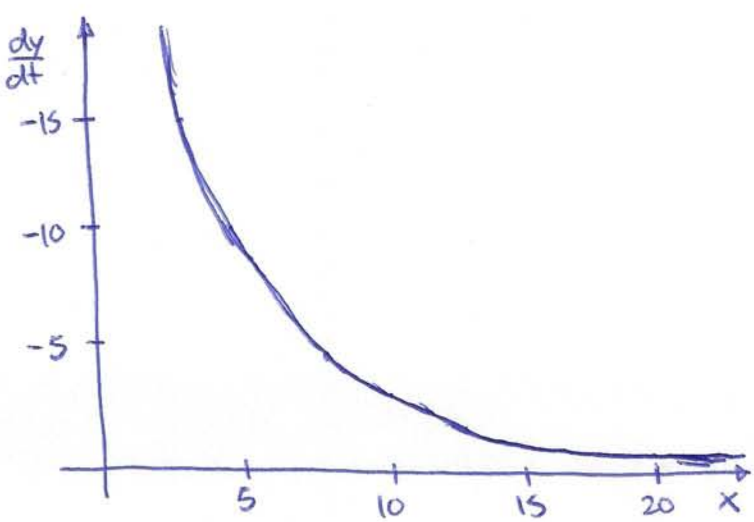
Solve for $\frac{dy}{dt}$ when $x = \ell - 8 = 20 - 8 = 12$ Feet, use eqn. (2)

$$\frac{\frac{dy}{dt}}{\ell} = -\frac{h}{x^2}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{\ell h}{x^2}\frac{dx}{dt} = -\frac{(20)(6)}{(12)^2}(2.5) = -2.08\bar{3} \frac{\text{Feet}}{\text{sec.}}$$

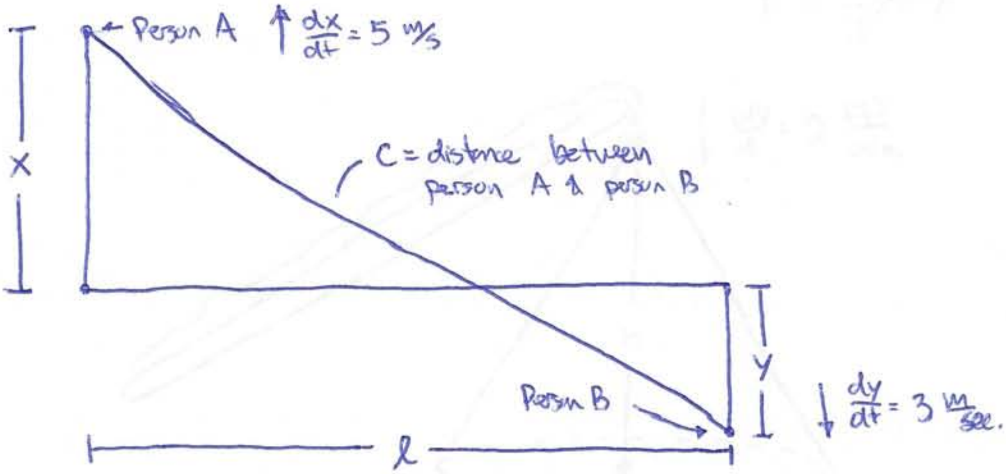
$$\boxed{\frac{dy}{dt} = -2 \frac{ft}{sec}}$$

Plot

$$\frac{dy}{dt} = -\frac{(20)(6)}{x^2}(2.5) = \frac{-300}{x^2}$$



Two people on bikes are separated by 350 meters. Person A starts riding north at a rate of 5 m/sec and Person B starts riding south at 3 m/sec. At what rate is the distance separating the two people changing 10 seconds after they start riding?



Pythagoras:

$$c^2 = \ell^2 + (x+y)^2 \qquad (1)$$

$\frac{d}{dt}\{$ 0 (ℓ = constant)

$$2c\frac{dc}{dt} = 2\ell\cancel{\frac{d\ell}{dt}} + 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \qquad (2)$$

Determine x, y & c after 10 seconds:

$$V_A = \left(\frac{dx}{dt}\right)_{ave.} = \frac{x}{t} \Rightarrow x = t\left(\frac{dx}{dt}\right)_{ave} = (10)(5) = 50\text{ m}$$

$$V_B = \left(\frac{dy}{dt}\right)_{ave} = \frac{y}{t} \Rightarrow y = t\left(\frac{dy}{dt}\right)_{ave.} = (10)(3) = 30\text{ m.}$$

equ. (1) $c^2 = \ell^2 + (x+y)^2$

$$c = \sqrt{\ell^2 + (x+y)^2}$$

$$= \sqrt{350^2 + (50+30)^2} = 359.03$$

Solve for $\frac{dc}{dt}$ when t=10 second (i.e. x=50, y=30 & C=359.03 m), equ. (2):

$$2c\frac{dc}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \Rightarrow \frac{dc}{dt} = \frac{x+y}{c}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{50+30}{359.03}(5+3) = 1.78\text{ m/s}$$

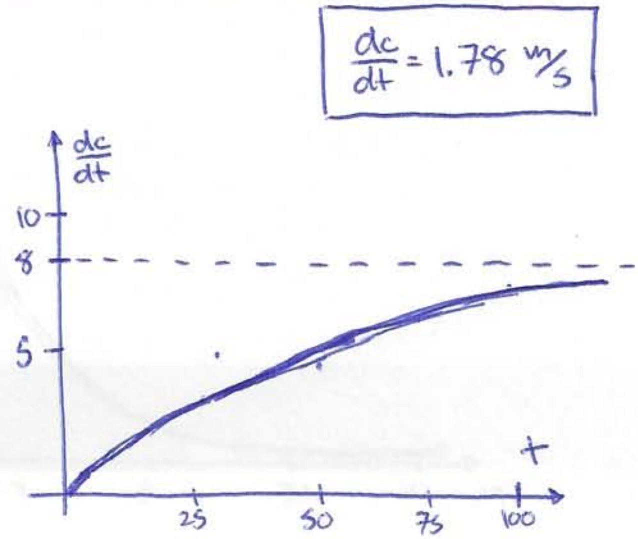
$$\boxed{\frac{dc}{dt} = 1.78\text{ m/s}}$$

sub velocity eqns & (1) into (2)

$$\frac{dc}{dt} = \frac{tx' + ty'}{\sqrt{(\ell)^2 + (ty' + tx')^2}}(x'+y') \stackrel{...}{=} \frac{\frac{dx}{dt} + \frac{dy}{dt}}{\sqrt{1 + \frac{\ell^2}{t^2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)^2}}}$$

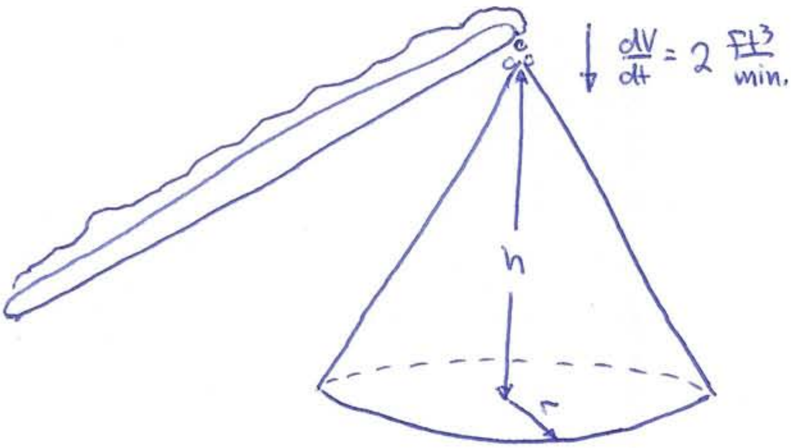
Plot

$$\frac{dc}{dt} = \frac{5+3}{\sqrt{1 + \frac{350^2}{t^2(5+3)^2}}} = \frac{8}{\sqrt{1 + \left(\frac{350}{8t}\right)^2}}$$



Sand is dumped off a conveyer belt into a pile at the rate of 2 ft³/min. The sand pile is shaped like a cone whose height and base diameter are always equal. At what rate is the height of the pile growing when the pile is 5 feet high? The volume of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$



Volume of cone:

$V = \frac{1}{3}\pi r^2 h$ (assume $h = 2r \Rightarrow r = \frac{1}{2}h$, given) (1)

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

$\frac{d}{dt}$

$\frac{dV}{dt} = \frac{3}{12}\pi h^2 \frac{dh}{dt}$ (2)

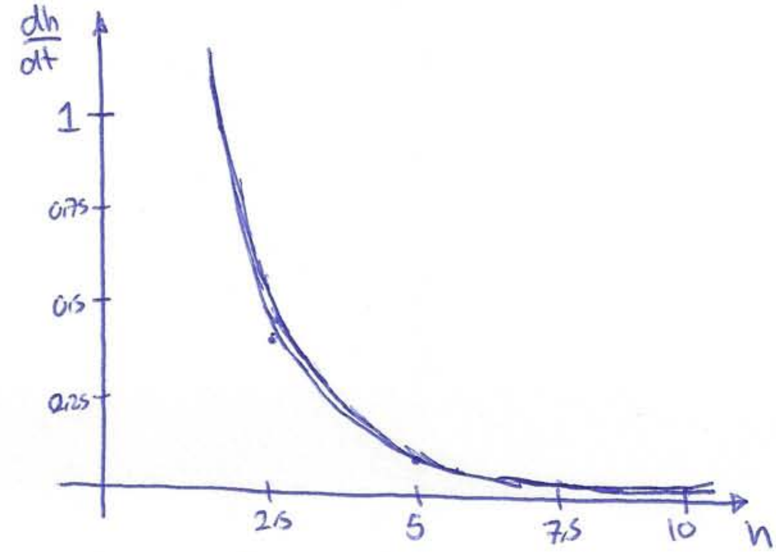
Solve for $\frac{dh}{dt}$ when $h = 5$ feet, equ. (2):

$\frac{dV}{dt} = \frac{3\pi}{12} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{3\pi h^2}\frac{dV}{dt} = \frac{12}{3\pi (5)^2}(2) = 0.1019 \frac{ft}{min}$

$$\boxed{\frac{dh}{dt} = 0.1 \frac{ft}{min}}$$

Plot:

$\frac{dh}{dt} = \frac{12}{3\pi h^2}(2) = \frac{8}{\pi h^2}$



Suppose that we have two resistors connected in parallel with resistances R1 and R2. The total resistance, R, is then given by:

$$\frac{1}{R} = \frac{1}{R1} + \frac{1}{R2}$$

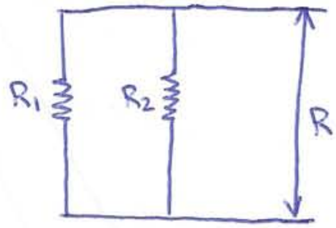
Suppose that R1 is increasing at a rate of 0.4 ohms/min and R2 is decreasing at a rate of 0.7 ohms/min. At what rate is R changing when R1=80 ohms and R2=100 ohms?

Given

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (1)$$

$\frac{d}{dt}$

$$\frac{-1}{R^2}\frac{dR}{dt} = \frac{-1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \qquad (2)$$



Solve for $\frac{dR}{dt}$ when $R_1 = 80$ & $R_2 = 100$, equ. (2)

solve for R, equ. (1)

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = 0.0225$$

$$R = \frac{1}{0.0225} = 44.4\ \Omega$$

$$-\frac{1}{R^2}\frac{dR}{dt} = \frac{-1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$$

$$\frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}\right)$$

$$= (44.4)^2\left(\frac{1}{80^2}(0.4) + \frac{1}{100^2}(-0.7)\right)$$

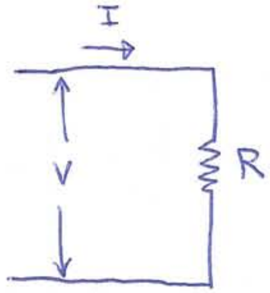
$$= -0.0148 \text{ ohms/sec.}$$

$$\boxed{\frac{dR}{dt} = -0.01 \text{ ohms/sec}}$$

Suppose that we have a circuit which has the current (I) passing through a resistor of resistance R. The voltage drop across the resistor, V, is given by ohm's Law:

$$V = IR$$

If the resistance of the resistor is increasing by 1 ohm/min and the current is decreasing at 0.1 amps/min what is the voltage drop across the resistor when the resistance is 10 ohms and the current is 1 amp?



Given

$$V = IR$$

$\frac{d}{dt}$

$$\frac{dV}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}$$

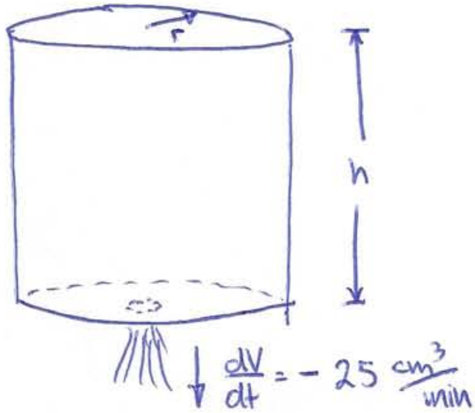
Solve for $\frac{dV}{dt}$ when $R = 10\,\Omega$ & $I = 1\,A$

$$\frac{dV}{dt} = (10)(-0.1) + (1)(1) = 0 \text{ volts/min}$$

$$\boxed{\frac{dV}{dt} = 0 \text{ volts/min}}$$

A cylindrical tank is filled with water. The tank stands upright and has a radius of 20 cm. How fast does the height of water in the tank drop when the water is being drained at 25 cm³/min? The volume of a cylinder is given by:

$$V = \pi r^2 h$$



Volume of cylinder:

$$V = \pi r^2 h \qquad (1)$$

$$\frac{d}{dt}\Big(\qquad \frac{dV}{dt} = 2\pi r h \cancel{\frac{dr}{dt}}^{\,0 \ (r = \text{constant})} + \pi r^2 \frac{dh}{dt} \qquad (2)$$

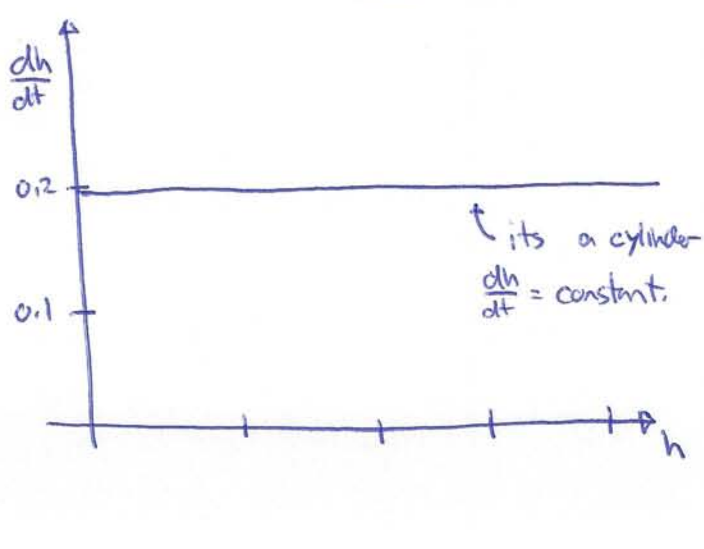
Solve for $\frac{dh}{dt}$, eqn. (1):

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{\pi (20)^2}(25) = 0.0198 \text{ cm/min}$$

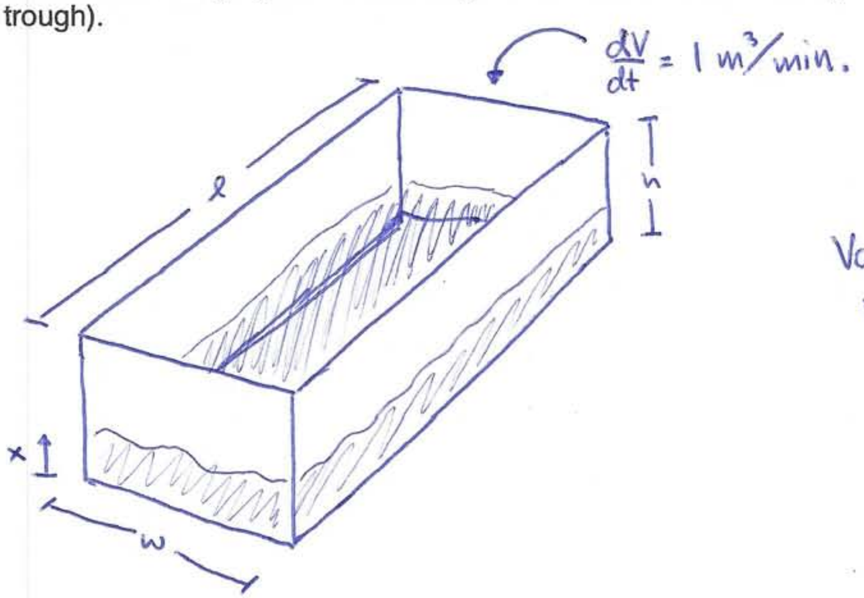
$$\boxed{\frac{dh}{dt} = 0.02 \text{ cm/min}}$$

Plot

$$\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt}$$



A trough is cut out of the ground in the shape of a rectangular box. The trough is 10 meters long, 4 meters wide and 2 meters deep. Rain is filling up the trough at a rate of 1.0 m^3/minute. How fast is the water level changing inside the trough when the water is half way up (1 meters from the bottom of the trough).



easy

Volume of water $= V = \ell w x$

$$V(x) = \ell w x = (10)(4)x = 40x$$

$\frac{d}{dt}$

does not depend on "x", it's a rectangle.

$$\frac{dV}{dt} = 40\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{40}\frac{dV}{dt} \qquad (1)$$

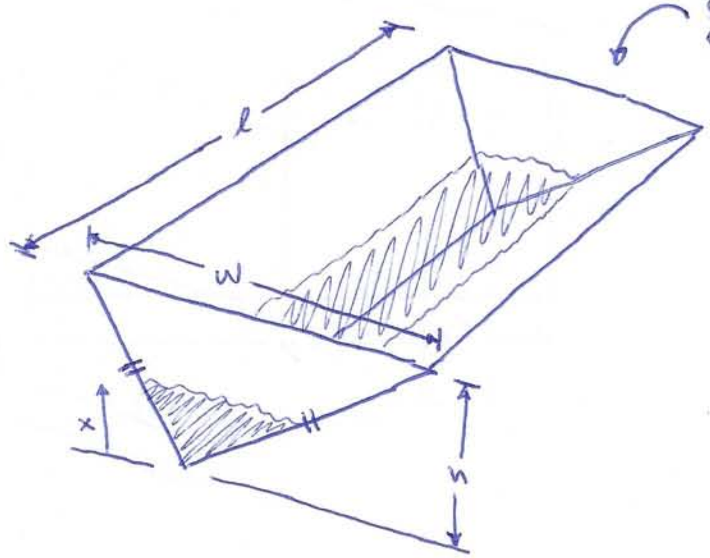
when the water is half way up $x = 1$, use equ. (1)

$$\frac{dx}{dt} = \frac{1}{40}(1) = 0.025 \text{ m/minute.} \qquad \boxed{\frac{dx}{dt} = 0.025 \text{ m/min.}}$$

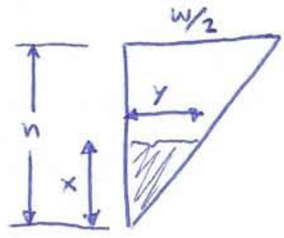
A trough is cut out of the ground in the shape of a prism where the ends are in the shape of isosceles triangles. The trough is 10 meters long, 4 meters wide and 2 meters deep. Rain is filling up the trough at a rate of 1.0 m³/minute. How fast is the water level changing inside the trough when the water is half way up (1 meters from the bottom of the trough).

$\frac{dV}{dt} = 1 \ m^3/min.$

hard



Similar triangles.



$$\frac{x}{y} = \frac{h}{(w/2)} \Rightarrow y = \frac{wx}{2h} \quad (1)$$

$$\text{Volume of water} = \tfrac{1}{2}(\text{height})(\text{base})(\text{lenght}) = \tfrac{1}{2}(x)(2y)(l) \quad (2)$$

Sub (1) into (2)

$$V(x) = \tfrac{1}{2}(x)\left[2\left(\frac{wx}{2h}\right)\right](l) = \frac{1}{2}\frac{wl}{h}x^2$$

$\frac{d}{dt}$

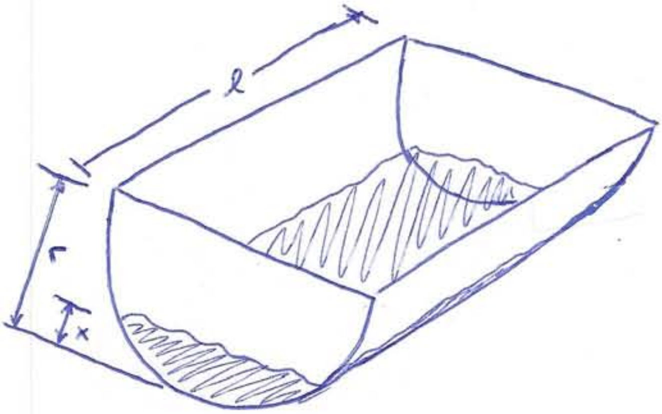
$$\frac{dV}{dt} = \frac{wl}{h} x \frac{dx}{dt} \quad (3)$$

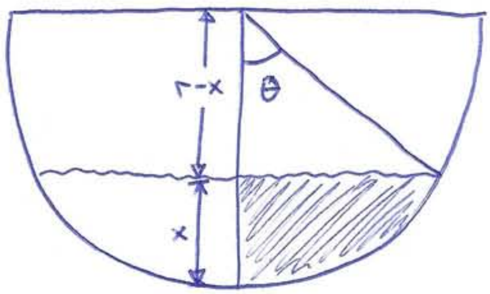
Solve for $\frac{dx}{dt}$ when water is half way up, $x = 1$, use equ (3).

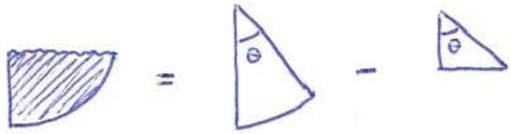
$$\frac{dx}{dt} = \frac{h}{wlx}\frac{dV}{dt} = \frac{2}{(4)(10)(1)}(1) = \frac{2}{40} = 0.05 \ m/min. \qquad \boxed{\frac{dx}{dt} = 0.05 \ m/min}$$

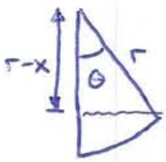
A trough is cut out of the ground in the shape of a half cylinder. The trough is 10 meters long and has a radius of 2 meters. Rain is filling up the trough at a rate of 1.0 m^3/minute. How fast is the water level changing inside the trough when the water is half way up (1 meter~~s~~ from the bottom of the trough).

very hard.

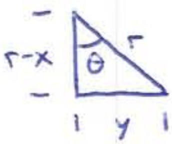






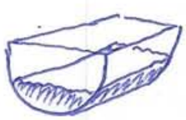


$$\theta = \cos^{-1}\left(\frac{r-x}{r}\right) \Rightarrow \text{Area} = \frac{\theta r^2}{2} = \frac{r^2}{2}\cos^{-1}\left(\frac{r-x}{r}\right)$$



$$y = \sqrt{r^2-(r-x)^2} \Rightarrow \text{Area} = \frac{(\text{base})(\text{height})}{2} = \frac{(r-x)\left[r^2-(r-x)^2\right]^{\frac{1}{2}}}{2}$$

$$\text{Area} = \frac{r^2}{2}\cos^{-1}\left(\frac{r-x}{r}\right) - \frac{1}{2}(r-x)\left[r^2-(r-x)^2\right]^{\frac{1}{2}}$$



$$\text{Volume} = 2l\left(\frac{r^2}{2}\cos^{-1}\left(\frac{r-x}{r}\right) - \frac{1}{2}(r-x)\left[r^2-(r-x)^2\right]^{\frac{1}{2}}\right)$$

$$\boxed{V(x) = lr^2\cos^{-1}\left(\frac{r-x}{r}\right) - l(r-x)\left[r^2-(r-x)^2\right]^{\frac{1}{2}}}$$

$$V(x) = lr^2\cos^{-1}\left(\frac{r-x}{r}\right) - l(r-x)\left[r^2-(r-x)^2\right]^{\frac{1}{2}}$$

$\frac{d}{dt}$

$$\frac{dV}{dt} = lr^2\left[-\frac{1}{\left(1-\left(\frac{r-x}{r}\right)^2\right)^{\frac{1}{2}}}\right]\left[-\frac{1}{r}\right]\left[\frac{dx}{dt}\right] - \left[(l)\left(-1\frac{dx}{dt}\right)\left(r^2-(r-x)^2\right)^{\frac{1}{2}} + l(r-x)\left(\frac{1}{2}\right)\left(r^2-(r-x)^2\right)^{-\frac{1}{2}}(2)(r-x)\left(-\frac{dx}{dt}\right)\right]$$

$$= \frac{lr}{\left(1-\left(\frac{r-x}{r}\right)^2\right)^{\frac{1}{2}}}\frac{dx}{dt} + (l)\left(r^2-(r-x)^2\right)^{\frac{1}{2}}\frac{dx}{dt} - \frac{l(r-x)^2}{\left(r^2-(r-x)^2\right)^{\frac{1}{2}}}\frac{dx}{dt}$$

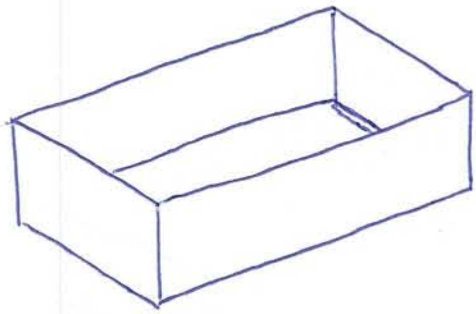
when the water is halfway up $x=1$, solve:

$$\frac{dV}{dt} = \frac{(10)(2)}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}}}\frac{dx}{dt} + (10)\left(2^2-(1)^2\right)^{\frac{1}{2}}\frac{dx}{dt} - \frac{(10)(1)^2}{\left(2^2-(1)^2\right)^{\frac{1}{2}}}\frac{dx}{dt}$$

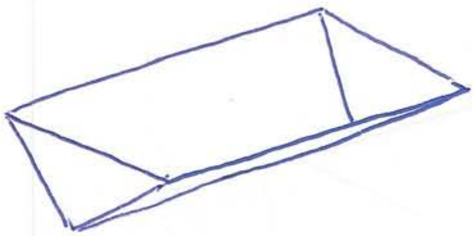
$$\frac{dV}{dt} = 23.1\frac{dx}{dt} + 17.3\frac{dx}{dt} - 5.8\frac{dx}{dt} = 34.6\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{34.6}\frac{dV}{dt} = \frac{1}{34.6}(1) = 0.029 \text{ m/min.}$$

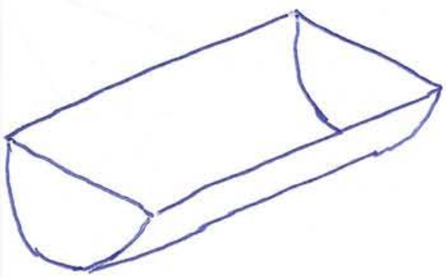
Compare



$$\frac{dV}{dt} = lw\frac{dx}{dt}$$



$$\frac{dV}{dt} = \frac{lwx}{h}\,\frac{dx}{dt}$$



$$\frac{dV}{dt} = \left[\frac{lr}{\left(1-\left(\frac{r-x}{r}\right)^2\right)^{\frac{1}{2}}} + l\left(r^2-(r-x)^2\right)^{\frac{1}{2}} - \frac{l(r-x)^2}{\left(r^2-(r-x)^2\right)^{\frac{1}{2}}}\right]\frac{dx}{dt}$$

Sub: $l=10,\ w=4\ (r=2),\ h=2;\ \frac{dV}{dt}=1$ and plot $\frac{dx}{dt}$ as a function of x (i.e. $\frac{dx}{dt}(x)$):