Derivatives of Select Functions

Polynomial (Power) Functions:

$$\boxed{F(x) = ax^n}$$

Binomial Theorem

$$F(x+\Delta x) = a(x+\Delta x)^n = a\left(x^n + nx^{n-1}\Delta x + \tfrac{1}{2}n(n-1)x^{n-2}\Delta x^2 + \ldots + \Delta x^n\right)$$

$$F(x+\Delta x) - F(x) = ax^n + anx^{n-1}\Delta x + \tfrac{1}{2}an(n-1)x^{n-2}\Delta x^2 + \ldots + a\Delta x^n - ax^n$$

$$\frac{d}{dx}\left(F(x)\right) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{anx^{n-1}\Delta x + \tfrac{1}{2}an(n-1)x^{n-2}\Delta x^2 + \ldots + a\Delta x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} anx^{n-1} + \tfrac{1}{2}an(n-1)x^{n-2}\Delta x + \ldots + a\Delta x^{n-1}$$

$$\boxed{\frac{d}{dx}(ax^n) = anx^{n-1}}$$

Trigonometry (sin) Function:

$$\boxed{f(x) = \sin(x)}$$

Identity

$$\frac{d}{dx}(\sin(x)) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x)\right) - \sin(x)}{\Delta x}$$

$$= \sin(x) \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

$$= \sin(x) \lim_{\Delta x \to 0} \frac{(\cos(\Delta x) - 1)(\cos(\Delta x) + 1)}{(\Delta x)(\cos(\Delta x) + 1)} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

identity

$$= \sin(x) \lim_{\Delta x \to 0} \frac{-\sin^2(\Delta x)}{\Delta x(\cos(\Delta x) + 1)} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

$$= \sin(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} \lim_{\Delta x \to 0} -\sin(\Delta x) \lim_{\Delta x \to 0} \frac{1}{\cos(\Delta x) + 1} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

limit proof   limit proof

$$= \sin(x) \left[\ 1\ \right]\left[\ -0\ \right]\left[\ \tfrac{1}{2}\ \right] + \cos(x)\left[\ 1\ \right]$$

$$\boxed{\frac{d}{dx}(\sin(x)) = \cos(x)}$$

Trigonometry (cos) Function:

$$\boxed{F(x) = \cos(x)}$$

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(x + \frac{\pi}{2}\right)\right) = \cos\left(x + \frac{\pi}{2}\right)(1+0) = \cos\left(x+\frac{\pi}{2}\right) = -\sin(x)$$

$$\boxed{\frac{d}{dx}(\cos(x)) = -\sin(x)}$$

Exponential and Logarithm Functions:

$$\boxed{F(x) = a^x}$$

$$\frac{d}{dx}(a^x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = f'(x)$$

$$f'(0) = a^0 \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

so $f'(x) = f'(0)\, f(x)$ — To find the derivative ($f'(x)$) we need to know the derivative @ $x=0$ ($f'(0)$)...

so we set $f'(0) = 1 = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$ — definition of the value "e"

$$\boxed{\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1}$$

in general

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln(a)}$$

$$\boxed{F(x) = \ln x}$$

$$\frac{d}{dx}(\ln x) = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{x}{x}\right)\left(\ln\left(\frac{x+\Delta x}{x}\right)\right) = \frac{1}{x} \lim_{\Delta x \to 0} \ln\left(\frac{x+\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \frac{1}{x}$$

in general

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$