Instructor:	Frank Secretain
Course:	Math 20
Date:	October 17, 2024
Assessment:	Test 2
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	4 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

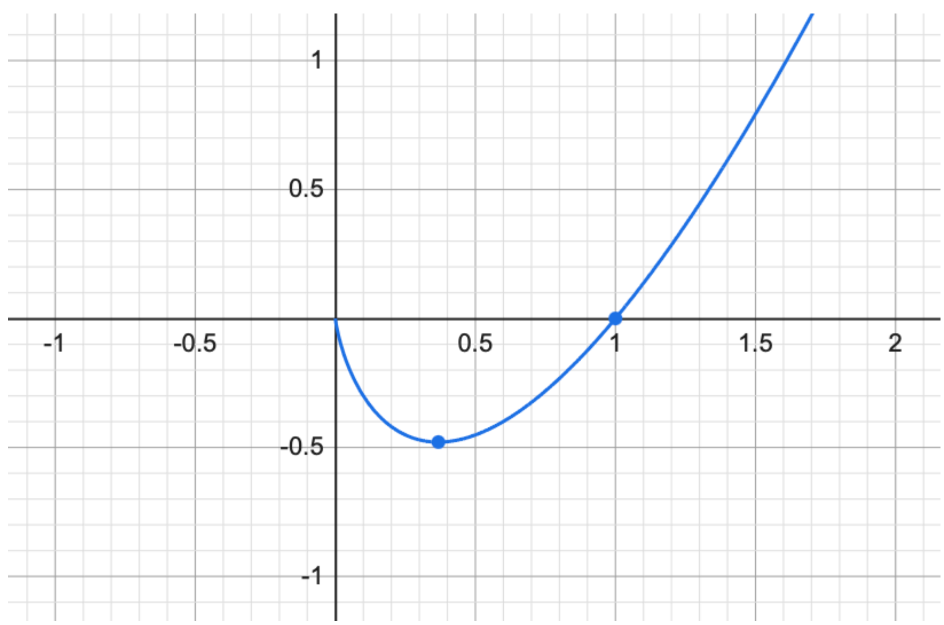
(5 marks) Determine the derivative with respect to x of the following equation.

$$y(x) = 4a^2x^2 \sin(2ax) + 2$$

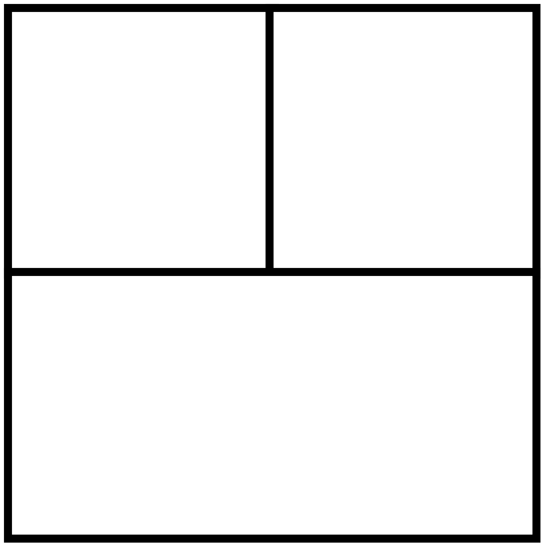
$$\frac{y^3}{3} - y^2x - y = x^2$$

(5 marks) Determine the tangent line at x=0.5 for the following function. Plot the tangent line on the plot.

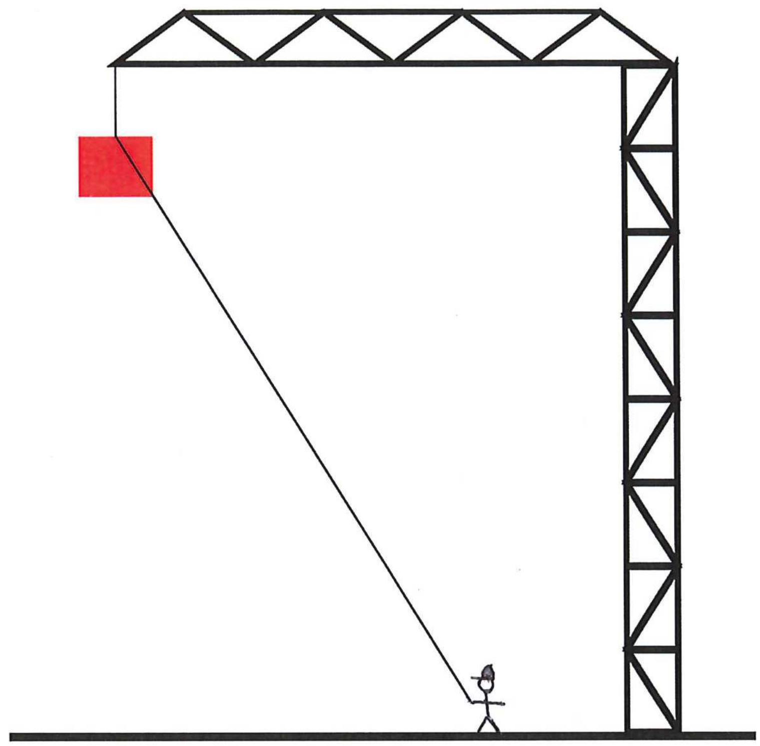
$$y = 3xlog(x)$$



(5 marks) Given 100 meters of fencing determine the maximum area of the proposed enclosure.



(5 marks) A construction crane is lifting a load straight up from the ground while a worker is standing 30 feet away from the point directly beneath the load. The crane is lifting the load at a rate of 5 feet per second. How fast does the worker have to release the rope attached to the load when the load is 40 feet above the ground?



(5 marks) Determine the derivative with respect to x of the following equation.

$$y(x) = 4a^2x^2\sin(2ax) + 2$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 8a^2x\sin(2ax) + 4a^2x^2\cos(2ax)(2a) + 0$$

$$\boxed{\frac{dy}{dx} = 8a^2x\sin(2ax) + 8a^2x^2\cos(2ax)}$$

$$\frac{y^3}{3} - y^2x - y = x^2$$

$\frac{d}{dx}$

$$y^2\frac{dy}{dx} - \left(2y\frac{dy}{dx}x + y^2\frac{dx}{dx}\right) - 1\frac{dy}{dx} = 2x\frac{dx}{dx}$$

$$y^2\frac{dy}{dx} - 2yx\frac{dy}{dx} - y^2 - \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left(y^2 - 2yx - 1\right) = 2x + y^2$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y^2}{y^2 - 2yx - 1}}$$

(5 marks) Determine the tangent line at x=0.5 for the following function. Plot the tangent line on the plot.

$$y = 3xlog(x)$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 3\log(x) + 3x\frac{1}{x\ln(10)}$$

$$= 3\log(x) + \frac{3}{\ln(10)}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 3\log\left(\tfrac{1}{2}\right) + \frac{3}{\ln(10)} = 0.3998$$

$$y\left(\tfrac{1}{2}\right) = 3\left[\tfrac{1}{2}\right]\log\left(\tfrac{1}{2}\right)$$

$$y\left(\tfrac{1}{2}\right) = -0.4514$$

line equation: $y = ax + b$

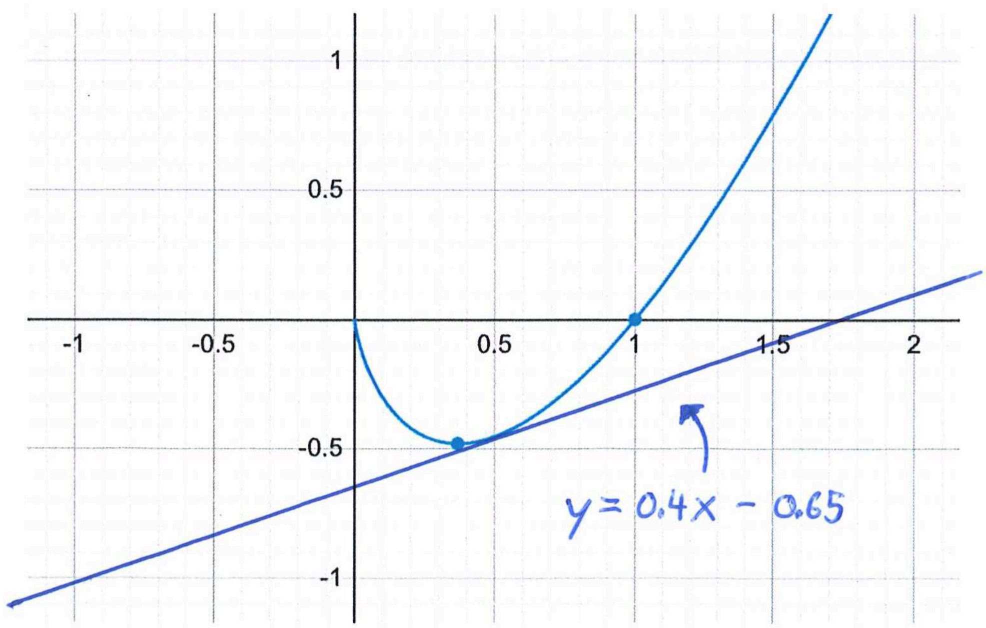
$$y = [0.3998]x + b$$

$$b = [-0.4514] - 0.3998\left[\tfrac{1}{2}\right]$$

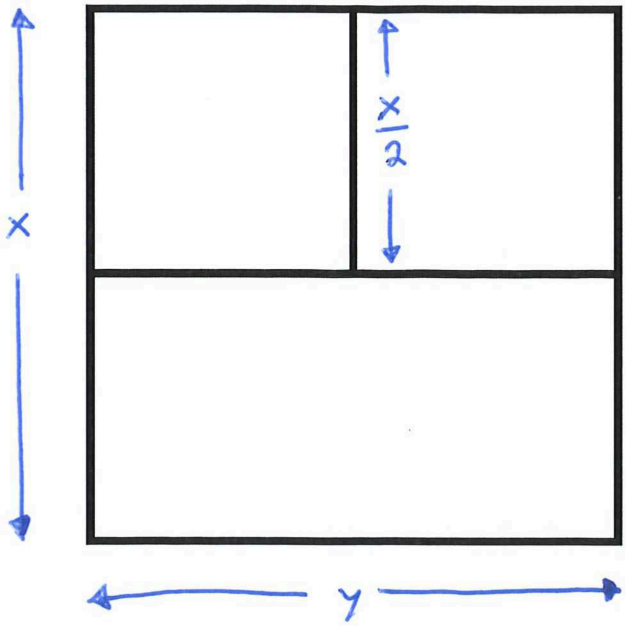
$$b = -0.6513$$

so

$$y = 0.4x - 0.65$$



(5 marks) Given 100 meters of fencing determine the maximum area of the proposed enclosure.



$$P = \frac{5}{2}x + 3y \quad (1)$$

$$A = xy \quad (2)$$

solve (1) for y:

$$y = \frac{P}{3} - \frac{5}{6}x \quad (1a)$$

sub (1a) into (2):

$$A = x\left[\frac{P}{3} - \frac{5}{6}x\right]$$

$$A = \frac{P}{3}x - \frac{5}{6}x^2$$

take derivative wrt x:

$$\frac{dA}{dx} = \frac{P}{3} - \frac{5}{3}x = 0 \ (\text{max/min})$$

so

$$x = \frac{P}{5} = \frac{100}{5} = 20$$

sub into (1a)

$$y = \frac{100}{3} - \frac{5}{6}(20)$$

$$y = 16\tfrac{2}{3} = 16.\overline{6}$$

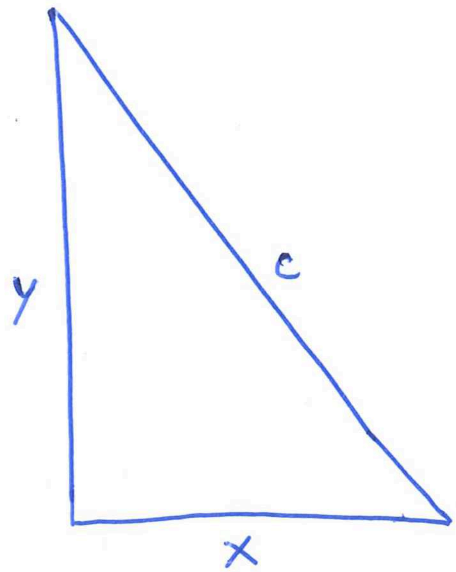
So

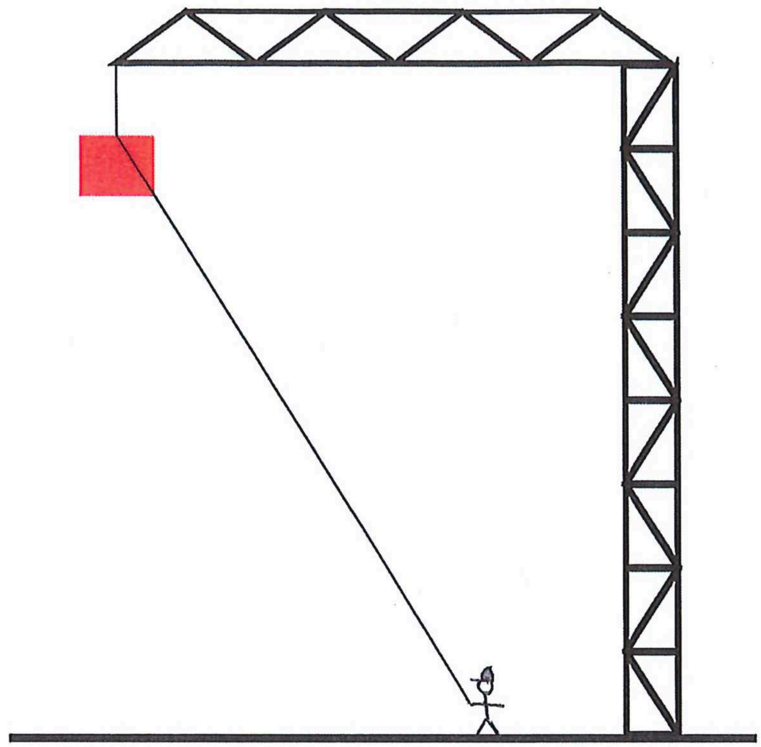
$$A = xy$$

$$= (20)(16.\overline{6})$$

$$\boxed{A = 333.3 \ m^2}$$

(5 marks) A construction crane is lifting a load straight up from the ground while a worker is standing 30 feet away from the point directly beneath the load. The crane is lifting the load at a rate of 5 feet per second. How fast does the worker have to release the rope attached to the load when the load is 40 feet above the ground?





$$c^2 = x^2 + y^2$$

$\frac{d}{dt}$

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad 0 \left(\frac{dx}{dt} = 0\right)$$

$$\frac{dc}{dt} = \frac{y}{c}\frac{dy}{dt}$$

@ $y = 40$

$$c = \sqrt{x^2 + y^2}$$

$$c = \sqrt{(30)^2 + (40)^2}$$

$$c = 50$$

and

$$\frac{dc}{dt} = \frac{40}{50}(5) = 4 \text{ ft/s}$$

$$\boxed{\frac{dc}{dt} = 4 \text{ feet/sec}}$$