Instructor:	Frank Secretain
Course:	Math 20
Date:	September 26, 2024
Assessment:	Test 1
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	5 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$ (exponentials)

$$\int ln(x)dx = xln(x) - x$$

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

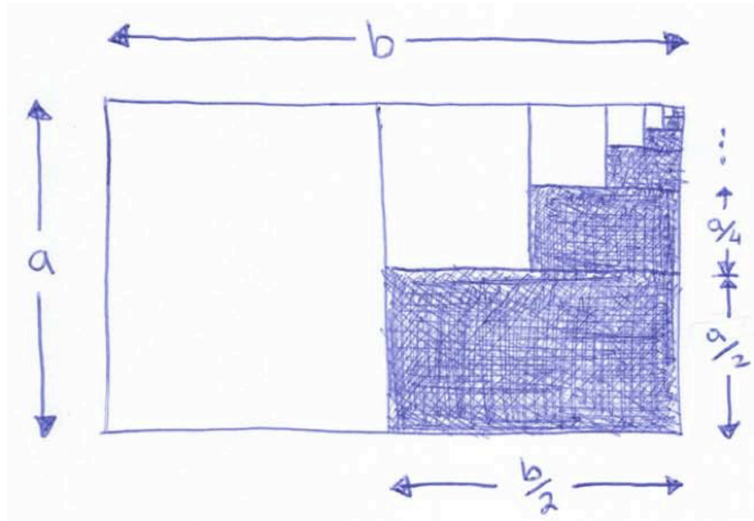
$$\int udv = uv - \int vdu$$

(2 marks each) Determine the 75th number in the sequence and the sum from the first number to the 75th number for each of the following series.

2, 2.2, 2.42, …

2, 2.2, 2.4, …

(4 marks) Use an infinite series to determine the area of the shaded area in terms of lengths of “a” and “b” given the below figure. Set up an infinite series by adding the appropriate areas and then sum to infinity.



(1 mark each) Determine the limits of the following expressions:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$$

$$\lim_{x \to \infty} \frac{3x^2 + 2x + 12}{2x^2 - 1000}$$

(2 marks) Take the derivative with respect to “x” of the following function **using the definition of the derivative**.

$$y(x) = 4x - 2$$

(2 marks each) Determine the derivative with respect to x of the following equation.

$$y(x) = 4x - 2$$

$$y(x) = \frac{\alpha\epsilon^2}{2}x^4 + \beta\cos(x) + 1$$

$$y(x) = \frac{\alpha\epsilon^2}{2}x^4\cos(x) + 1$$

$$y(x) = \frac{\alpha\epsilon^2}{2} 4^x \cos(\gamma x^2 - x) + 1$$

$$y(x) = \frac{\alpha\epsilon^2}{2} e^{\cos(\gamma x^2 - x)} + 1$$

(2 marks each) Determine the 75th number in the sequence and the sum from the first number to the 75th number for each of the following series.

2, 2.2, 2.42, …

$K = \begin{matrix} 2.2 - 2 = 0.2 \\ 2.42 - 2.2 = 0.22 \end{matrix}$ ✗

$r = \begin{matrix} 2.2/2 = 1.1 \\ 2.42/2.2 = 1.1 \end{matrix}$ ✓ (geometric)

$a_1 = 2$

$a_n =$

$n = 75$

$r = 1.1$

$S_n =$

$a_n = a_1 r^{n-1}$

$= 2 (1.1)^\wedge(75-1)$

$a_{75} = 2312.5$

$S_n = a_1 \frac{1 - r^n}{1 - r}$

$= 2 \frac{1 - (1.1)^\wedge 75}{1 - 1.1}$

$S_{75} = 25417.9$

2, 2.2, 2.4, …

$K = \begin{matrix} 2.2 - 2 = 0.2 \\ 2.4 - 2.2 = 0.2 \end{matrix}$ ✓ (arithmetic)

$r = \begin{matrix} 2.2/2 = 1.1 \\ 2.4/2.2 = 1.09 \end{matrix}$ ✗

$a_1 = 2$

$a_n =$

$n = 75$

$K = 0.2$

$S_n =$

$a_n = a_1 + (n-1)K$

$= 2 + (75-1)(0.2)$

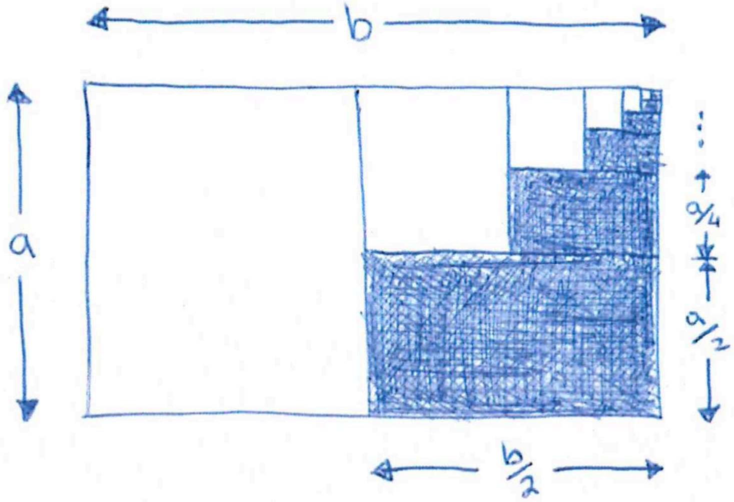
$a_{75} = 16.8$

$S_n = \frac{n}{2}(a_1 + a_n)$

$= \frac{75}{2}(2 + 16.8)$

$S_{75} = 705$

(4 marks) Use an infinite series to determine the area of the shaded area in terms of lengths of "a" and "b" given the below figure. Set up an infinite series by adding the appropriate areas and then sum to infinity.



$$\left(\frac{a}{2}\right)\left(\frac{b}{2}\right), \left(\frac{a}{4}\right)\left(\frac{b}{4}\right), \left(\frac{a}{8}\right)\left(\frac{b}{8}\right), \ldots$$

$$= \frac{ab}{4}, \frac{ab}{16}, \frac{ab}{64}, \ldots$$

$$= ab\left(\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\right)$$

geometric

$$k = \frac{1}{16} - \frac{1}{4} = -\frac{3}{16}$$
$$\frac{1}{64} - \frac{1}{16} = -\frac{3}{64} \quad \times$$

$$r = \frac{1}{16} / \frac{1}{4} = \frac{1}{4}$$
$$\frac{1}{64} / \frac{1}{16} = \frac{1}{4} \quad \checkmark \text{ (geometric)}$$

$a_1 = \frac{1}{4}$

$a_n =$

$n \Rightarrow \infty$

$r = \frac{1}{4}$

$S_n =$

$$a_n = a_1 r^{n-1}$$
$$= \frac{1}{4}\left(\frac{1}{4}\right)^{\infty} \to 0$$
$$a_\infty = 0$$

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$
$$= \frac{1}{4} \cdot \frac{1 - \left(\frac{1}{4}\right)^{\infty}}{1 - \left(\frac{1}{4}\right)}$$
$$S_\infty = \frac{1}{3}$$

$$\boxed{\text{Sum} = \frac{ab}{3}}$$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm 3}{2} = 1, -2$

(1 mark each) Determine the limits of the following expressions:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x-(-2))}{(x-1)} = x + 2 = \boxed{3}$$

or

x	0.9	0.99	0.999
y	2.9	2.99	2.999

x	1.1	1.01	1.001
y	3.1	3.01	3.001

$\to \boxed{3}$

$$\lim_{x \to \infty} \frac{3x^2 + 2x + 12}{2x^2 - 1000} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \frac{3 + \frac{2}{x} + \frac{12}{x^2}}{2 - \frac{1000}{x^2}} = \boxed{\frac{3}{2}}$$

or

x	10^3	10^6	10^9
y	1.502	1.500	1.500

$\to \boxed{1.5}$

(2 marks) Take the derivative with respect to "x" of the following function **using the definition of the derivative.**

$$y(x) = 4x - 2$$

$y(x) = 4x - 2$

$y(x+\Delta x) = 4(x+\Delta x) - 2$

$= 4x + 4\Delta x - 2$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[4x + 4\Delta x - 2] - [4x - 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 2 - 4x + 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x} = \boxed{4}$$

(2 marks each) Determine the derivative with respect to x of the following equation.

$$y(x) = 4x - 2$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 4 + 0$$

$$y(x) = \frac{\alpha\epsilon^2}{2}x^4 + \beta\cos(x) + 1$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{4\alpha\epsilon^2}{2}x^3 - \beta\sin(x) + 0$$

$$y(x) = \frac{\alpha\epsilon^2}{2}x^4\cos(x) + 1$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{4\alpha\epsilon^2}{2}x^3\cos(x) - \frac{\alpha\epsilon^2}{2}x^4\sin(x) + 0$$

$$y(x) = \frac{\alpha\epsilon^2}{2} 4^x \cos(\gamma x^2 - x) + 1$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{\alpha\epsilon^2}{2} 4^x \ln(4) \cos(\gamma x^2 - x) - \frac{\alpha\epsilon^2}{2} 4^x \sin(\gamma x^2 - x)(2\gamma x - 1) + 0$$

$$y(x) = \frac{\alpha\epsilon^2}{2} e^{\cos(\gamma x^2 - x)} + 1$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{\alpha\epsilon^2}{2} e^{\cos(\gamma x^2 - x)} \cancel{\ln(e)}^{1} \left(-\sin(\gamma x^2 - x)(2\gamma x - 1)\right) + 0$$