Instructor:	Frank Secretain
Course:	Math 20
Date:	December 12, 2024
Assessment:	Final Test
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	4 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks) A road construction crew is paving a new highway. In the first hour, they lay 50 meters of asphalt. Each subsequent hour, they increase their productivity by 10 meters (e.g. 60 meters in the second hour, 70 meters in the third hour, and so on). How many hours will it take them to pave a 10 kilometer stretch of highway?

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 4x^{\frac{1}{3}} + \cos(x)$$

$$y(x) = 4x^{\frac{1}{3}} \cos(x)$$

$$y(x) = \frac{ab^2}{2} \sin(a^2 x^3 + x - 1) + \frac{3}{x^b} + 1$$

$$y^2 - yx + 3 = x^2$$

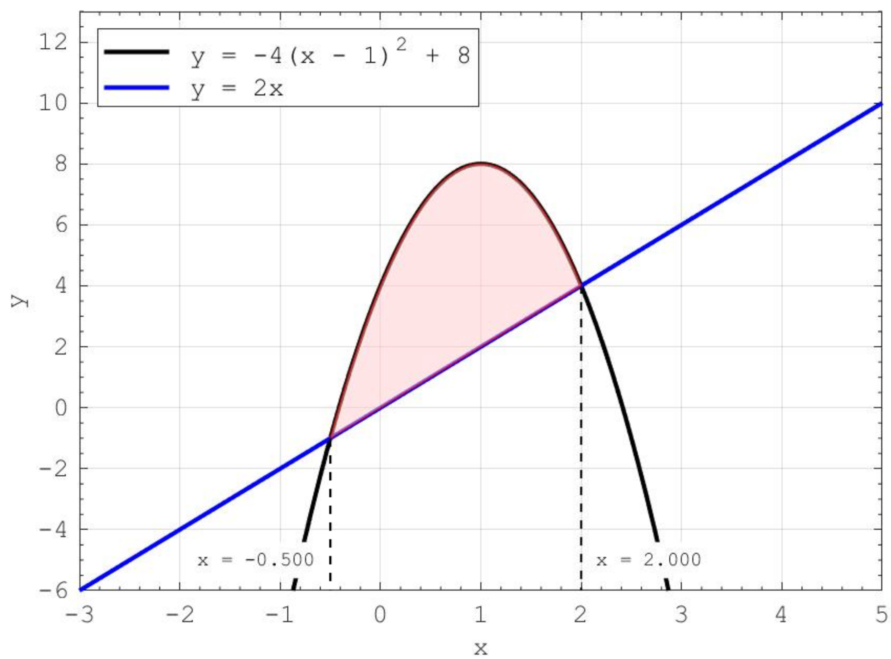
(2 marks each) Integrate with respect to "x" the following functions.

$$\int ax^4 - b\cos(x) - 1 \ dx$$

$$\int \eta\cos(2^a x + 1) - 1 \ dx$$

$$\int \frac{ax^2}{b(2x^3+1)+c} + 4 \; dx$$

(4 marks) Find the area.



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50, 60, 70, ...

$k = \begin{matrix} 60-50 = 10 \\ 70-60 = 10 \end{matrix}$ ✓

$r = \begin{matrix} 60/50 = 1.2 \\ 70/60 = 1.1\overline{6} \end{matrix}$ ×

$a_1 = 50$ m

$a_n =$

$K = 10$

$n =$

$S_n = 10000$ m

$a_n = a_1 + (n-1)K$ (1)

$S_n = \frac{n}{2}(a_1 + a_n)$ (2)

sub (1) into (2)

$S_n = \frac{n}{2}(a_1 + [a_1 + (n-1)k])$

$= \frac{n}{2}(2a_1 + kn - k)$

$2S_n = 2a_1 n + kn^2 - kn$

$0 = kn^2 + (2a_1 - k)n - 2S_n$

$0 = 10n^2 + (2 \cdot 50 - 10)n - 2(10000)$

$0 = 10n^2 + 90n - 20000$

$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-90 \pm \sqrt{(-90)^2 - 4(10)(-20000)}}{2(10)}$

$= -4.5 \pm 44.947$

$= 40.45, -49.45$

$n = 41$ hours

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 4x^{\frac{1}{3}} + \cos(x)$$

$\frac{d}{dx}$ ↓

$$\frac{dy}{dx} = \frac{4}{3} x^{-\frac{2}{3}} - \sin(x)$$

$$y(x) = 4x^{\frac{1}{3}} \cos(x)$$

$\frac{d}{dx}$ ↓

$$\frac{dy}{dx} = \frac{4}{3} x^{-\frac{2}{3}} \cos(x) - 4x^{\frac{1}{3}} \sin(x)$$

$$y(x) = \frac{ab^2}{2}\sin(a^2x^3 + x - 1) + \frac{3}{x^b} + 1$$

$$y(x) = \frac{ab^2}{2}\sin(a^2x^3 + x - 1) + 3x^{-b} + 1$$

$\frac{d}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{ab^2}{2}\cos(a^2x^3 + x - 1)(3a^2x^2 + 1) - 3bx^{-b-1}}$$

$$y^2 - yx + 3 = x^2$$

$\frac{d}{dx}$

$$2y\frac{dy}{dx} - \frac{dy}{dx}x - y = 2x$$

$$\frac{dy}{dx}(2y - x) = 2x + y$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y}{2y - x}}$$

(2 marks each) Integrate with respect to "x" the following functions.

$$\int ax^4 - b\cos(x) - 1 \; dx$$

$$\frac{a}{5}x^5 - b\sin(x) - x + c$$

$$\int \eta\cos(2^a x + 1) - 1 \; dx$$

$$\frac{\eta}{2^a}\sin\left(2^a x + 1\right) - x + c$$

$$\int \frac{ax^2}{b(2x^3+1)+c} + 4\ dx$$

$$\int \frac{ax^2}{b(2x^3+1)+c}\,dx + \int 4\,dx$$

let $u = 2x^3 + 1$

$\frac{du}{dx} = 6x^2$

$dx = \frac{du}{6x^2}$

$$\int \frac{a\cancel{x^2}}{bu+c}\,\frac{du}{6\cancel{x^2}} + \int 4\,dx$$

$$\frac{a}{6}\int \frac{1}{bu+c}\,du + 4x + c$$

let $w = bu + c$

$\frac{dw}{du} = b$

$du = \frac{dw}{b}$

$$\frac{a}{6}\int \frac{1}{w}\,\frac{dw}{b} + 4x + c$$

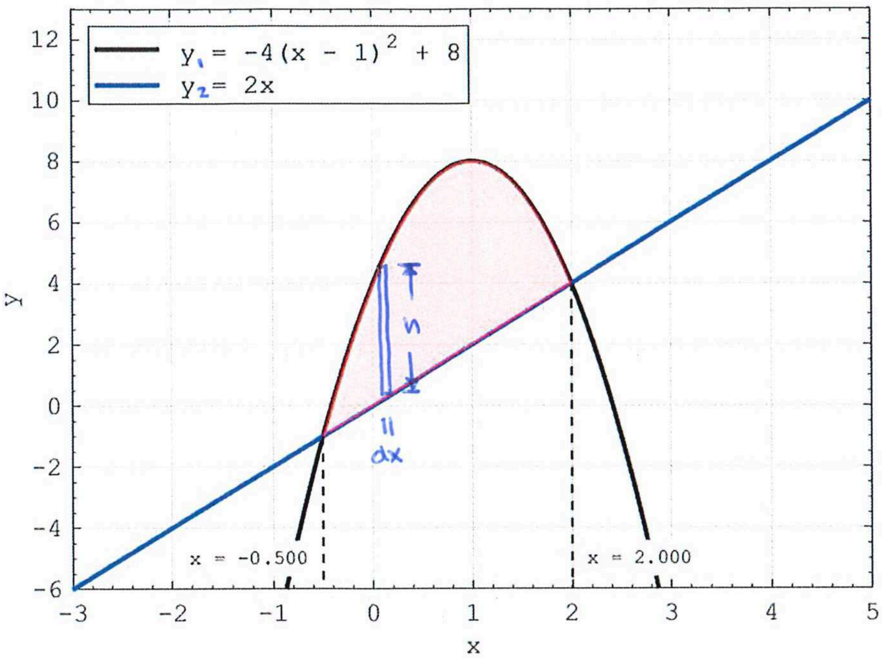
$$\frac{a}{6b}\int w^{-1}\,dw + 4x + c$$

$$\frac{a}{6b}\ln(|w|) + 4x + c$$

$$\frac{a}{6b}\ln(|bu+c|) + 4x + c$$

$$\boxed{\frac{a}{6b}\ln\left(\left|b(2x^3+1)+c\right|\right) + 4x + c}$$

(4 marks) Find the area.



$$\int dA = \int h \, dx$$

$$A = \int_{-0.5}^{2} (y_1 - y_2) \, dx$$

$$= \int_{-0.5}^{2} \left(-4(x-1)^2 + 8 - 2x\right) dx$$

$$= \int_{-0.5}^{2} -4(x-1)^2 - 2x + 8 \quad dx$$

$$= \left[-\frac{4}{3}(x-1)^3 - x^2 + 8x \right]_{-0.5}^{2}$$

$$= \left[-\frac{4}{3}([2]-1)^3 - [2]^2 + 8[2] \right] - \left[-\frac{4}{3}([-0.5]-1)^3 - [-0.5]^2 + 8[-0.5] \right]$$

$$= \left[-\frac{4}{3} - 4 + 16 \right] - \left[\frac{9}{2} - \frac{1}{4} - 4 \right]$$

$$= \left[\frac{32}{3} \right] - \left[\frac{1}{4} \right] = \boxed{\frac{125}{12} = 10.41\overline{6}}$$