Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 1
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	5 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k = \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

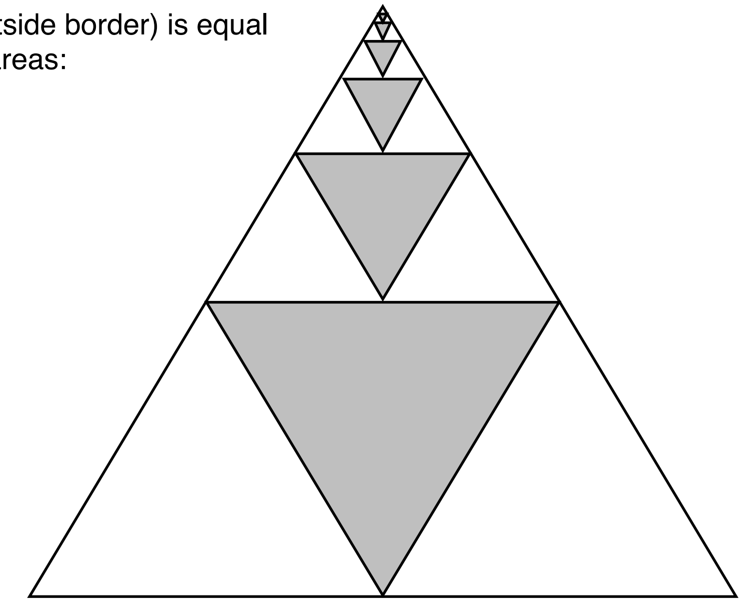
(2 marks each) Determine the 75th number in the sequence and the sum from the first number to the 75th number for each of the following series.

5, 5.05, 5.1005, …

5, 5.05, 5.10, …

(4 marks) It can be argued that after driving 1 million kilometres you are concerned an expert driver. Suppose, on your first year of driving you drive 5000 km and for every year after you drive 1000 km more than the previous year (i.e. on the second year you drive 6000 km and the third 7000 km). How many years of driving must you drive to achieve 1 million kilometres.

(4 marks) The area of the biggest triangle (the outside border) is equal to one. Calculate the sum of the infinite shaded areas:



(4 marks) Determine the limits of the following expressions:

$$\lim_{x \to 0} \frac{4x^2 + 2}{3x - 1}$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$$

$$\lim_{x \to \infty} \frac{3x^2 + 2x + 12}{2x^2 - 1000}$$

(4 marks) Take the derivative with respect to “x” of the following function **using the definition of the derivative**.

$$y(x) = 2x$$

$$y(x) = ax + 4x - c$$

(2 marks each) Determine the 75th number in the sequence and the sum from the first number to the 75th number for each of the following series.

5, 5.05, 5.1005, …

arithmetic ?

$5.05 - 5 = 0.05$

$5.1005 - 5.05 = 0.0505$ ×

geometric ?

$\frac{5.05}{5} = 1.01$ ✓ $r = 1.01$

$\frac{5.1005}{5.05} = 1.01$

$a_1 = 5$

$a_n =$

$r = 1.01$

$n = 75$

$S_n =$

$a_n = a_1 r^{n-1}$

$= (5)(1.01^{75-1})$

$= 10.44$

$a_{75} = 10.44$

$S_n = a_1 \frac{1 - r^n}{1 - r}$

$= (5) \frac{1 - 1.01^{75}}{1 - 1.01}$

$= 554.56$

$S_n = 554.56$

5, 5.05, 5.10, …

arithmetic ?

$5.05 - 5 = 0.05$

$5.10 - 5.05 = 0.05$ ✓ $k = 0.05$

$a_1 = 5$

$a_n =$

$K = 0.05$

$n = 75$

$S_n =$

$a_n = a_1 + (n-1)k$

$= 5 + (75-1)(0.05)$

$= 8.7$

$a_n = 8.7$

$S_n = \frac{n(a_1 + a_n)}{2}$

$= \frac{75(5 + 8.7)}{2}$

$= 513.75$

$S_n = 513.75$

(4 marks) It can be argued that after driving 1 million kilometres you are concerned an expert driver. Suppose, on your first year of driving you drive 5000 km and for every year after you drive 1000 km more than the previous year (i.e. on the second year you drive 6000 km and the third 7000 km). How many years of driving must you drive to achieve 1 million kilometres.

5000, 6000, 7000, ...

arithmetic ?

6000 − 5000 = 1000 ✓ $k = 1000$

7000 − 6000 = 1000

$a_1 = 5000$

$a_n =$

$k = 1000$

$n =$

$S_n = 1\,000\,000$

$a_n = a_1 + (n-1)k$

$a_n = 5000 + (n-1)1000$

$a_n = 1000n + 4000$ (1)

$S_n = \frac{n(a_1 + a_n)}{2}$

$10^6 = \frac{n(5000 + a_n)}{2}$

$2\times10^6 = 5000n + a_n n$ (2)

sub (1) into (2)

$2\times10^6 = 5000n + [1000n + 4000]n$

$2\times10^6 = 1000n^2 + 9000n$

$1000n^2 + 9000n - 2\times10^6 = 0$

$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

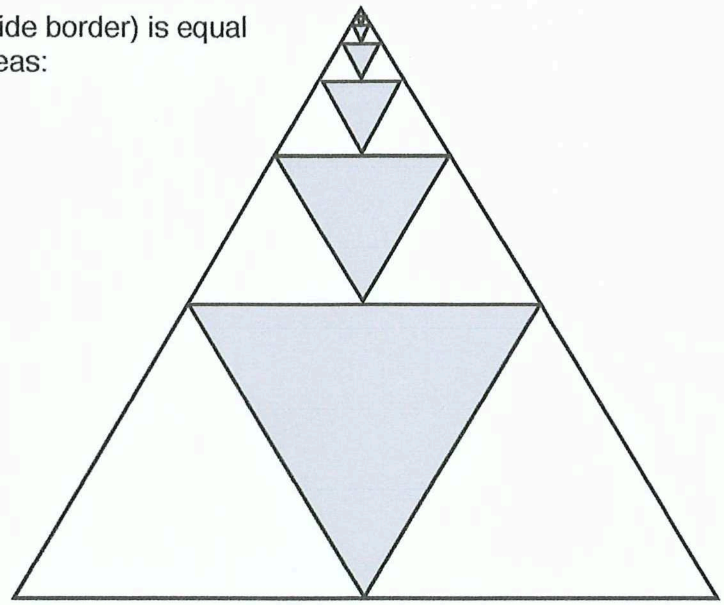
$= \frac{-9000 \pm \sqrt{(9000)^2 - 4(1000)(-2\times10^6)}}{2(1000)}$

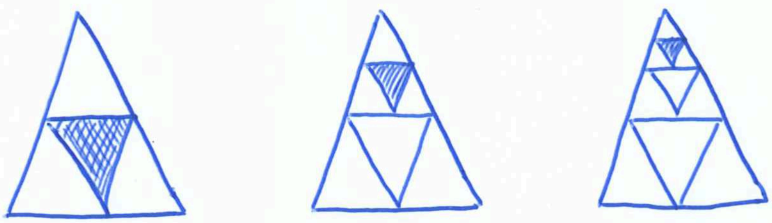
$= -4.5 \pm 44.9$

$= 40.4, -49.4$

\# of years = 40.4

(4 marks) The area of the biggest triangle (the outside border) is equal to one. Calculate the sum of the infinite shaded areas:





$\frac{1}{4}$, $\frac{1}{4^2}$, $\frac{1}{4^3}$, ...

0.25 , 0.0625 , 0.015625 , ...

geometric

$\frac{0.0625}{0.25} = 0.25$

$\frac{0.015625}{0.0625} = 0.25$

✓ $r = 0.25$

$S_n = a_1 \frac{1 - r^n}{1 - r}$

$= \left(\frac{1}{4}\right) \frac{1 - \left(\frac{1}{4}\right)^{\infty} \to 0}{1 - \left(\frac{1}{4}\right)}$

$= \frac{1}{3}$

$$\boxed{S_\infty = \frac{1}{3}}$$

(4 marks) Determine the limits of the following expressions:

$$\lim_{x\to 0} \frac{4x^2+2}{3x-1} = \frac{4(0)^2+2}{3(0)-1} = \frac{2}{-1} = -2$$

$$\boxed{\lim_{x\to 0} \frac{4x^2+2}{3x-1} = -2}$$

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \frac{(1)^2+(1)-2}{(1)-1} = \frac{0}{0}$$

$$\lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x\to 1} x+2 = (1)+2 = 3$$

$$\boxed{\lim_{x\to 1} \frac{x^2+x-2}{x-1} = 3}$$

$$\lim_{x\to\infty} \frac{3x^2+2x+12}{2x^2-1000} = \frac{3(\infty)^2+2(\infty)+12}{2(\infty)^2-1000} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{3x^2+2x+12}{2x^2-1000}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\infty} \frac{3+\frac{2}{x}+\frac{12}{x^2}}{2-\frac{1000}{x^2}} = \frac{3+\frac{2}{\infty}^{\,0}+\frac{12}{\infty^2}^{\,0}}{2-\frac{1000}{\infty^2}^{\,0}} = \frac{3}{2}$$

$$\boxed{\lim_{x\to\infty} \frac{3x^2+2x+12}{2x^2-1000} = \frac{3}{2}}$$

(4 marks) Take the derivative with respect to "x" of the following function **using the definition of the derivative**.

$$y(x) = 2x$$

$$y(x+\Delta x) = 2(x+\Delta x) = 2x + 2\Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2x + 2\Delta x] - [2x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$$

$$\boxed{\frac{dy}{dx} = 2}$$

$$y(x) = ax + 4x - c$$

$$y(x+\Delta x) = a(x+\Delta x) + 4(x+\Delta x) - c$$

$$= ax + a\Delta x + 4x + 4\Delta x - c$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[ax + a\Delta x + 4x + 4\Delta x - c] - [ax + 4x - c]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a\Delta x + 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0} a + 4 = a + 4$$

$$\boxed{\frac{dy}{dx} = a + 4}$$