Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Final Test
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	5 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$
$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$
$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$
$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$
$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$
$$\frac{d}{dx}(cos(x)) = -sin(x)$$
$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$
$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \left\{ \begin{array}{ll} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{array} \right\}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$
$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)
$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$
$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks) A 20,020 litre water tank is being filled. In the first minute, it receives 10 litres of water and each minute after the amount increases by 5 litres, i.e. the third minute is receives 20 litres of water. How long will it take to fill the tank?

(2 marks each) Take the derivative with respect to “x” of the following functions.

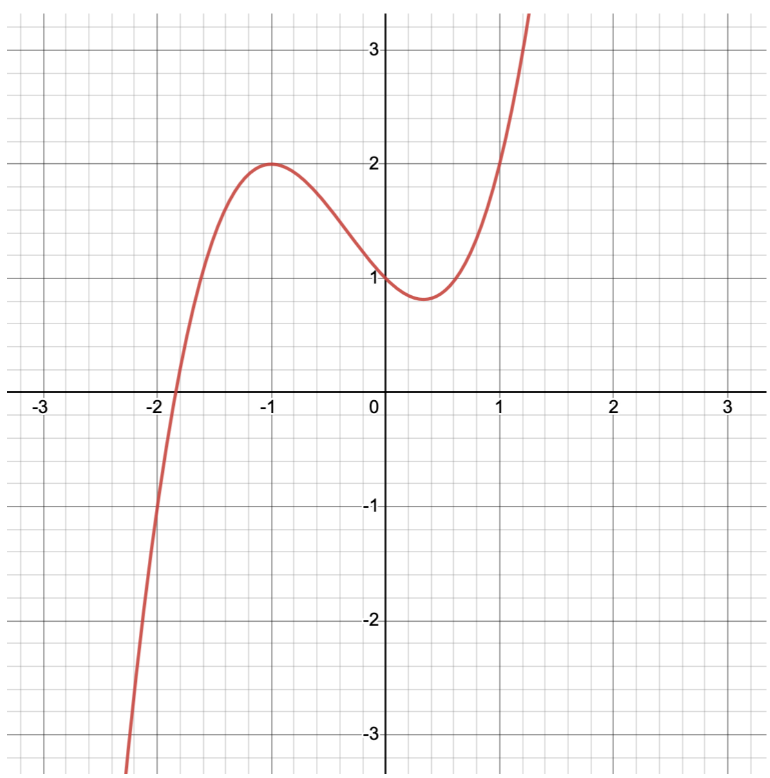
$$y(x) = 3x^{\frac{2}{3}} + \sin(2x)$$

$$y(x) = 3x^{\frac{2}{3}} \sin(2x)$$

$$y(x) = \frac{\alpha \sin(3x^2 - x + c)}{b} + \frac{4}{\sqrt{x}}$$

(3 marks) Determine the tangent line at x=1.0 of the below equation, draw the tangent line on the figure.

$$y = x^3 + x^2 - x + 1$$



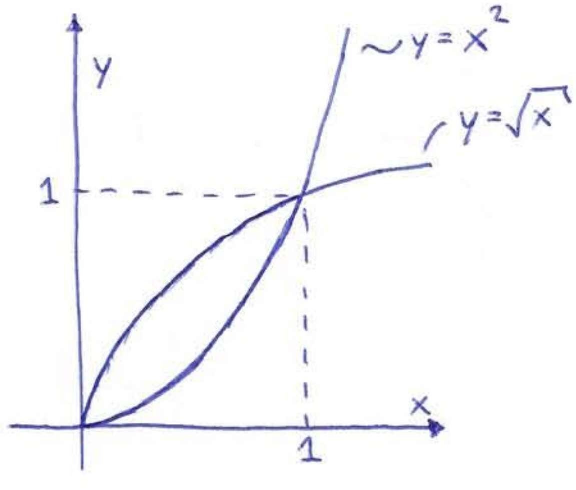
(2 marks each) Integrate with respect to “x” the following functions.

$$\int 3x^2 + 2\sin(x) - 1 \ dx$$

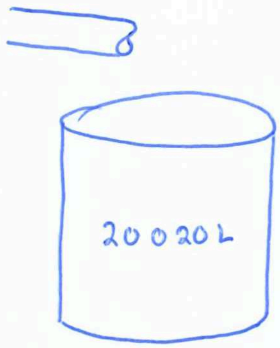
$$\int \beta \sin(ax + 2) + 2 \ dx$$

$$\int \frac{2x^2}{3(x^3+1)^{\frac{3}{2}}} + 1 \; dx$$

(3 marks) Find the area enclosed by the two functions.



(2 marks) A 20,020 litre water tank is being filled. In the first minute, it receives 10 litres of water and each minute after the amount increases by 5 litres, i.e. the third minute is receives 20 litres of water. How long will it take to fill the tank?



$10, 15, 20, 25, \dots$

$k = 15 - 10 = 5$ ✓
$20 - 15 = 5$

$r = 15/10 = 1.5$
$= 20/15 = 1.\bar{3}$ ×

arithemetic

$a_1 = 10$

$a_n =$

$n =$

$K = 5$

$S_n = 20020$

$a_n = a_1 + (n-1)K$

$a_n = 10 + (n-1)5$

$a_n = 10 + 5n - 5$

$a_n = 5 + 5n \quad (1)$

$S_n = \frac{n}{2}(a_1 + a_n)$

$20020 = \frac{n}{2}(10 + a_n)$

$20020 = 5n + a_n\frac{n}{2} \quad (2)$

sub (1) into (2)

$20020 = 5n + [5 + 5n]\frac{n}{2}$

$20020 = 5n + \frac{5}{2}n + \frac{5}{2}n^2$

$40040 = 10n + 5n + 5n^2$

$5n^2 + 15n - 40040 = 0$

$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= 88, -91$

$n = 88$ step

$= 88$ minutes

$\boxed{n = 88 \text{ minutes}}$

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 3x^{\frac{2}{3}} + \sin(2x)$$

$$\boxed{\frac{dy}{dx} = 2x^{-\frac{1}{3}} + 2\cos(2x)}$$

$$y(x) = 3x^{\frac{2}{3}}\sin(2x)$$

$$\boxed{\frac{dy}{dx} = \left(2x^{-\frac{1}{3}}\right)\left(\sin(2x)\right) + \left(3x^{\frac{2}{3}}\right)\left(2\cos(2x)\right)}$$

$$y(x) = \frac{\alpha \sin(3x^2 - x + c)}{b} + \frac{4}{\sqrt{x}}$$

$$y(x) = \frac{\alpha}{b} \sin\left(3x^2 - x + c\right) + \frac{4}{x^{\frac{1}{2}}}$$

$$y(x) = \frac{\alpha}{b} \sin\left(3x^2 - x + c\right) + 4x^{-\frac{1}{2}}$$

$$\boxed{\frac{dy}{dx} = \frac{\alpha}{b} \cos\left(3x^2 - x + c\right)\left(6x - 1\right) - 2x^{-\frac{3}{2}}}$$

(3 marks) Determine the tangent line at x=1.0 of the below equation, draw the tangent line on the figure.

$$y = x^3 + x^2 - x + 1$$

$$y\Big|_{x=1} = (1)^3 + (1)^2 - (1) + 1 = 2$$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

$$\frac{dy}{dx}\Big|_{x=1} = 3(1)^2 + 2(1) - 1 = 4$$

line

$$y = ax + b$$

$$a = \frac{dy}{dx}\Big|_{x=1} = 4$$

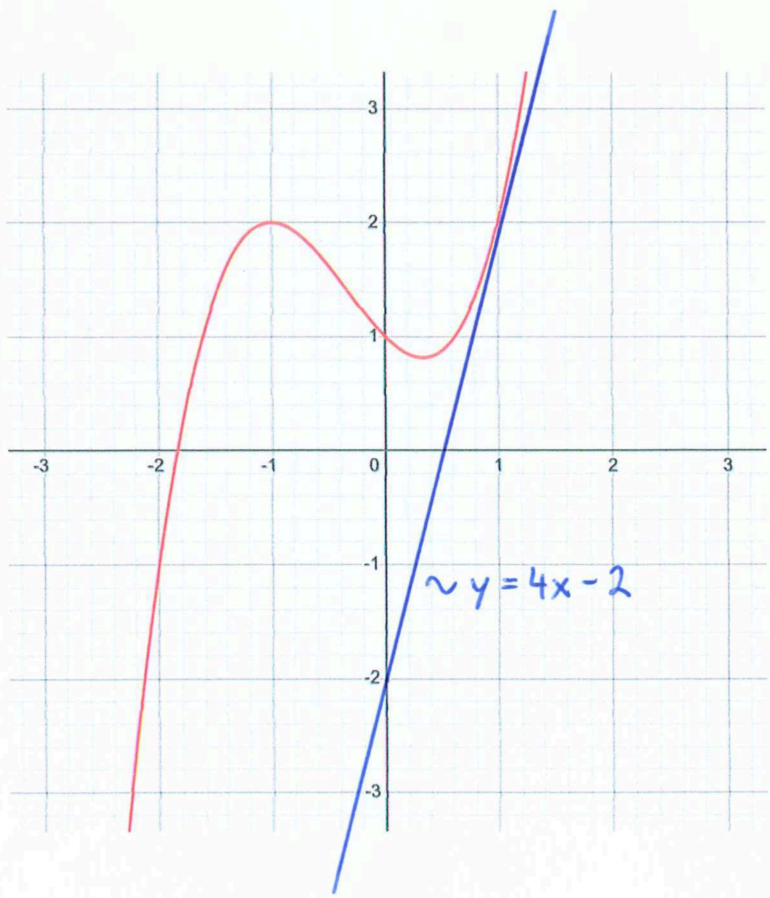
$$y = 4x + b$$

$$\boxed{y = 4x - 2}$$

sub $x=1$, $y=2$

$$[2] = 4[1] + b$$

$$b = -2$$



(2 marks each) Integrate with respect to "x" the following functions.

$$\int 3x^2 + 2\sin(x) - 1 \; dx$$

$$\boxed{x^3 - 2\cos(x) - x + C}$$

$$\int \beta \sin(ax+2) + 2 \; dx$$

let $u = ax + 2$

$$\frac{du}{dx} = a$$

$$dx = \frac{du}{a}$$

so

$$\int \beta \sin(u) + 2 \left(\frac{du}{a}\right)$$

$$-\frac{\beta}{a}\cos(u) + \frac{2}{a}u$$

$$-\frac{\beta}{a}\cos(ax+2) + \frac{2}{a}(ax+2)$$

$$-\frac{\beta}{a}\cos(ax+2) + 2x + \frac{4}{a}$$

$$\boxed{-\frac{\beta}{a}\cos(ax+2) + 2x + C}$$

$$\int \frac{2x^2}{3(x^3+1)^{\frac{3}{2}}} + 1 \, dx$$

let $u = x^3 + 1$

$$\frac{du}{dx} = 3x^2$$

so $dx = \frac{du}{3x^2}$

$$\int \frac{2\cancel{x^2}}{3u^{\frac{3}{2}}} \left(\frac{du}{3\cancel{x^2}}\right) + \int 1 dx$$

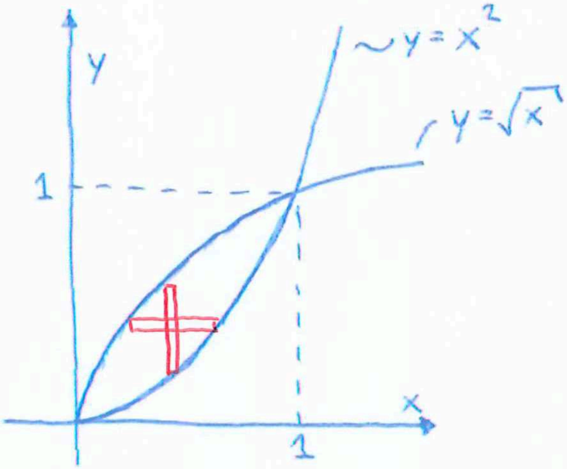
$$\int \frac{4}{9} u^{-\frac{3}{2}} du + \int dx$$

$$-\frac{4}{9}\frac{2}{1} u^{-\frac{1}{2}} + x + C$$

$$-\frac{8}{9}\left(x^3+1\right)^{-\frac{1}{2}} + x + C$$

$$\boxed{-\frac{8}{9\sqrt{x^3+1}} + x + C}$$

(3 marks) Find the area enclosed by the two functions.



$x = \sqrt{y}$

$x = y^2$

columns

$dA = \left(\sqrt{x} - x^2\right) dx$

$A = \int_0^1 x^{\frac{1}{2}} - x^2 \, dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$

$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right] - \left[0\right]$

$= \frac{1}{3}$

rows

$dA = \left(\sqrt{y} - y^2\right) dy$

$A = \int_0^1 y^{\frac{1}{2}} - y^2 \, dy$

$= \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{3}y^3\right]_0^1$

$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right] - \left[0\right]$

$= \frac{1}{3}$

$$\boxed{A = \frac{1}{3} = 0.\overline{3}}$$