Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 2
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	4 questions worth 25 marks + 1 bonus worth 2 marks.
Percentage of final grade:	25% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = (sec(x))^2$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$ (exponentials)

$$\int ln(x)dx = xln(x) - x$$

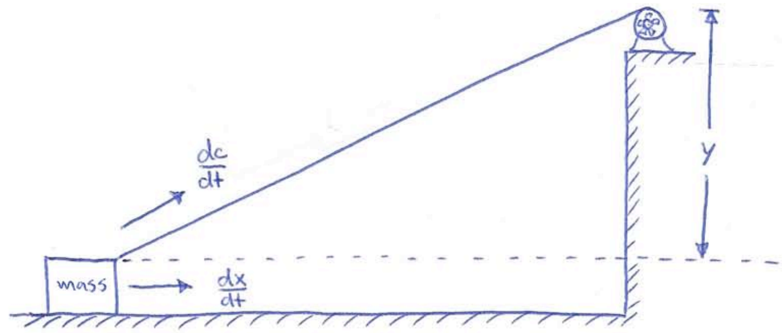
Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(5 marks) A mass is pulled horizontally along the ground by a rope attached to a motor which is 10 meter above the horizontal, as shown in the figure. If the rope is being pulled at a rate of 0.5 m/s how fast is the mass sliding along the horizontal ground when the mass is 20 meters from the vertical wall?



(5 marks) A 2 meter piece of wire is cut into two pieces, one piece is bent into a square and the other is bent into a circle. Where should the wire be cut so that the total area enclosed by both the square and circle are maximum?

Note: The circumference and area are $2\pi r$ and πr^2, respectively.

(10 marks) Evaluate the following integral:

$$\int 2x^2 dx$$

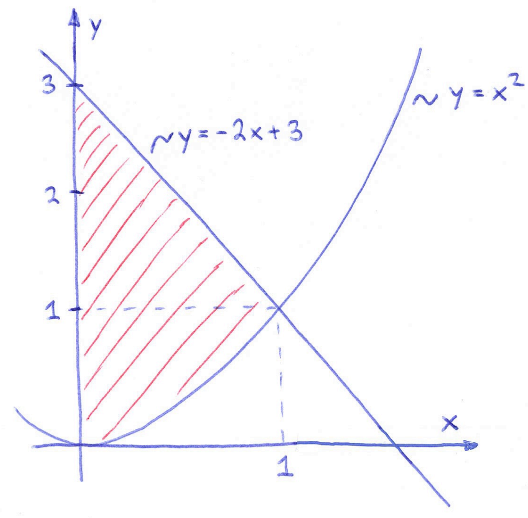
$$\int 5(2-x)^3 dx$$

$$\int 4cos(2x+1)dx$$

$$\int a(x+1)^n dx$$

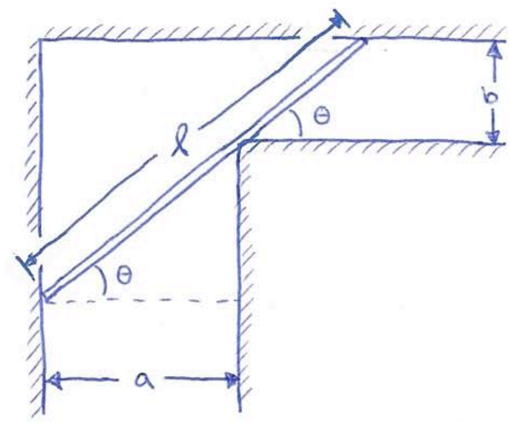
$$\int \frac{8}{3}x^3\sqrt{x^4+4}\ dx$$

(5 marks) Find the area of the shaded section.



(2 marks) BONUS:

A piece of pipe is being carried down a hallway that is 10 feet wide (a=10). At the end of the hallway there is a right-angle turn and the hallway narrows down to 8 feet wide (b=8). What is the longest pipe that can be carried (always keeping it horizontal to the ground) around the turn in the hallway?



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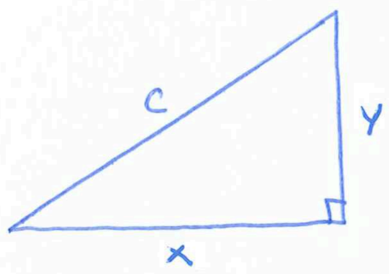
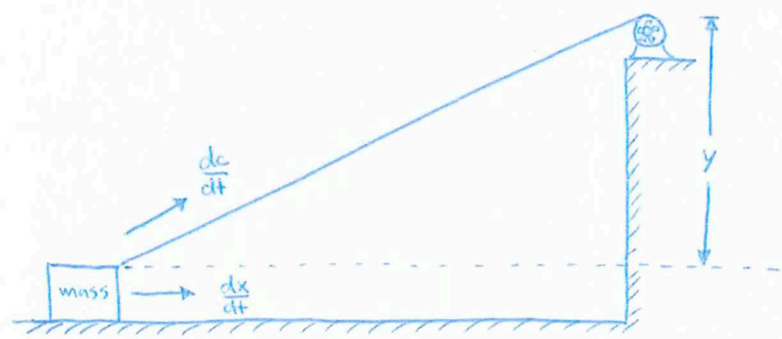




so

$$c^2 = x^2 + y^2$$

$\frac{d}{dt}$

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$ ($\frac{dy}{dt} \to 0$, y = constant)

$$c\frac{dc}{dt} = x\frac{dx}{dt}.$$

$$\frac{dx}{dt} = \frac{c}{x}\frac{dc}{dt}$$

at $x = 20$

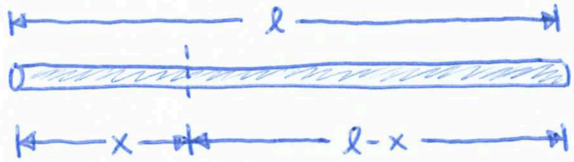
$$c = \sqrt{x^2 + y^2} = \sqrt{(20)^2 + (10)^2} = 22.4 \text{ m}$$

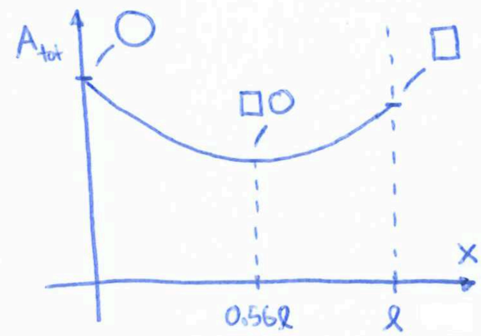
$$\frac{dx}{dt} = \frac{22.4}{20}(-0.5) = -0.56 \text{ m/s}$$

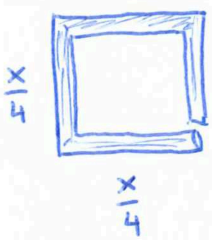
$$\boxed{\frac{dx}{dt} = -0.56 \text{ m/s}}$$

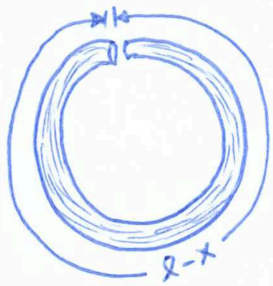
(5 marks) A 2 meter piece of wire is cut into two pieces, one piece is bent into a square and the other is bent into a circle. Where should the wire be cut so that the total area enclosed by both the square and circle are ~~maximum~~ minimum?

Note: The circumference and area are $2\pi r$ and πr^2, respectively.









$$C = 2\pi r$$
$$= \ell - x$$

so

$$r = \frac{\ell - x}{2\pi}$$

$$A = \left(\frac{x}{4}\right)^2 \qquad A = \pi\left(\frac{\ell - x}{2\pi}\right)^2$$

$$A_{total} = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{\ell - x}{2\pi}\right)^2$$

$$\frac{dA_{total}}{dx} = 2\left(\frac{x}{4}\right)\left(\frac{1}{4}\right) + 2\pi\left(\frac{\ell - x}{2\pi}\right)\left(\frac{-1}{2\pi}\right) = 0$$

minimum

$$\frac{x}{8} + \frac{x}{2\pi} - \frac{\ell}{2\pi} = 0$$

$$x\pi + 4x - 4\ell = 0$$

$$x(4 + \pi) = 4\ell$$

$$x = \frac{4}{4+\pi}\ell \cong 0.56\ell$$

For $\ell = 2$

$x = 1.12\,m$

(10 marks) Evaluate the following integral:

$$= \int 2x^2 dx$$

$$= \boxed{\frac{2}{3}x^3 + C}$$

$$= \int 5(2-x)^3 dx$$

let $u = 2 - x$

$\frac{du}{dx} = -1$

$dx = -du$

$$= \int 5u^3(-du)$$

$$= -\frac{5}{4}u^4 + c$$

$$= \boxed{-\frac{5}{4}(2-x)^4 + c}$$

$$= \int 4cos(2x+1)dx$$

let $u = 2x + 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$$= \int 4\cos(u)\frac{du}{2}$$

$$= 2\sin(u) + c$$

$$= \boxed{2\sin(2x+1) + c}$$

$$= \int a(x+1)^n dx$$

let $u = x+1$

$$\frac{du}{dx} = 1$$

$$dx = du$$

$$= \int a u^n \, du$$

$$= \frac{a}{n+1} u^{n+1} + c$$

$$\boxed{= \frac{a}{n+1}(x+1)^{n+1} + c}$$

$$= \int \frac{8}{3} x^3 \sqrt{x^4+4} \, dx$$

let $u = x^4 + 4$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

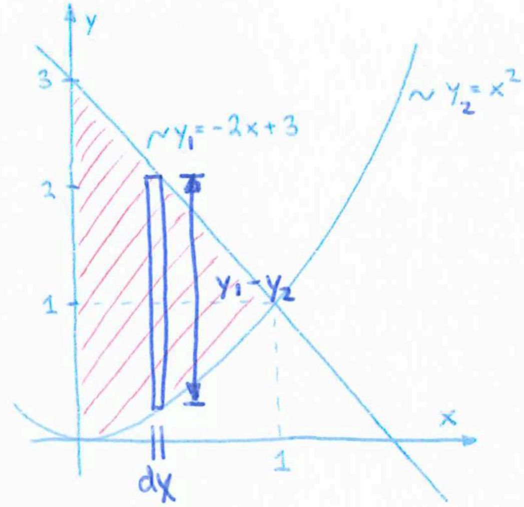
$$= \int \frac{8}{3} \cancel{x^3} (u)^{\frac{1}{2}} \frac{du}{4\cancel{x^3}}$$

$$= \int \frac{2}{3} u^{\frac{1}{2}} \, du$$

$$= \frac{2}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + c$$

$$\boxed{= \frac{4}{9}(x^4+4)^{\frac{3}{2}} + c}$$

(5 marks) Find the area of the shaded section.



$$A = \int_0^1 y_1 - y_2 \, dx$$

$$= \int_0^1 (-2x+3) - (x^2) \, dx$$

$$= \int_0^1 -2x + 3 - x^2 \, dx$$

$$= \left[-x^2 + 3x - \frac{1}{3}x^3 \right]_0^1$$

$$= \left[-(1)^2 + 3(1) - \frac{1}{3}(1)^3 \right] - \left[-(0)^2 + 3(0) - \frac{1}{3}(0)^3 \right]$$

$$= -1 + 3 - \frac{1}{3}$$

$$\boxed{A = \frac{5}{3} = 1.\bar{6}}$$

(2 marks) BONUS:

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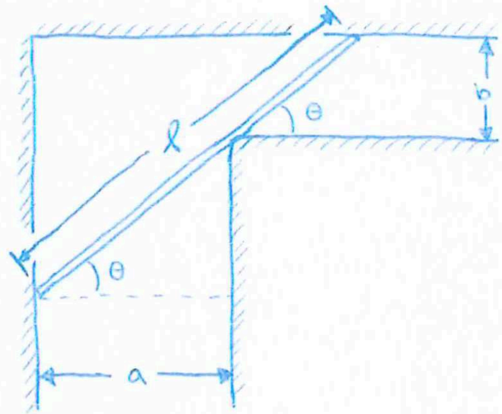
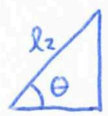
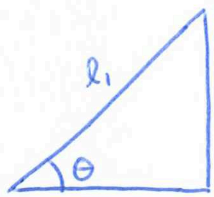


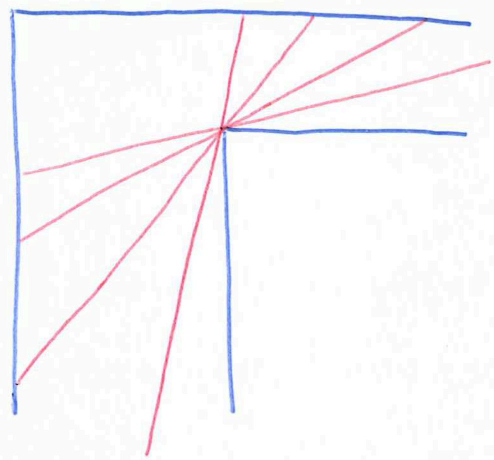


$$\sin\theta = \frac{b}{l_2}$$



$$\cos\theta = \frac{a}{l_1}$$

determine relationship between l & θ



$$l = l_1 + l_2$$

$$\boxed{l = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}}$$

maximum
minimum

$$\frac{dl}{d\theta} = -\frac{a}{\cos^2\theta}(-\sin\theta) - \frac{b}{\sin^2\theta}(\cos\theta) = 0$$

$$a\frac{\sin\theta}{\cos^2\theta} = b\frac{\cos\theta}{\sin^2\theta}$$

$$a\sin^3\theta = b\cos^3\theta$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{b}{a}$$

$$\tan^3\theta = \frac{b}{a}$$

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\theta = \tan^{-1}\left(\sqrt[3]{\frac{b}{a}}\right)$$

$$= \tan^{-1}\left(\sqrt[3]{\frac{8}{10}}\right)$$

$$\theta = 42.9^\circ$$

$$l = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

$$= \frac{10}{\cos(42.9)} + \frac{8}{\sin(42.9)}$$

$$\boxed{l = 25.4 \text{ Ft}}$$