Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Final Exam
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	8 questions worth 30 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \left\{ \begin{matrix} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{matrix} \right\}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$ (exponentials)

$$\int ln(x)dx = xln(x) - x$$

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks) Determine the 101th number in the sequence and the sum from the first number to the 101th number for each of the following series.

23, 29, 35, 41, ….

(2 marks) When you buy a car you could assume that each following year the value of the car will only be 80% of the previous year's value. If you bought a car for 20000$ how much would it be worth in 48 years, assume that the car lasts hat long… it a Tesla!!

(2 marks each) Take the derivative with respect to “x” of the following functions.

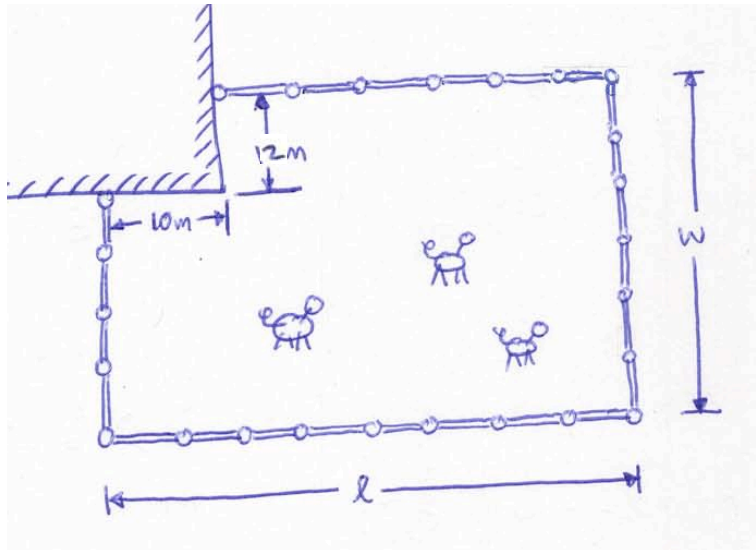
$$y(x) = 3x^{\frac{2}{3}} + \sin(2x)$$

$$y(x) = 3x^{\frac{2}{3}} \sin(2x)$$

(3 marks) Take the derivative with respect to "x" of the following function

$$y(x) = \left[(x^2+1)^3(x^{\frac{3}{2}}-1)+1\right]^2$$

(5 marks) You have to build a fence keeping a 10 m x 12 m section that will not be fenced, as shown in the diagram. If you are given 100 m of fencing what dimension would it have to maximize area?

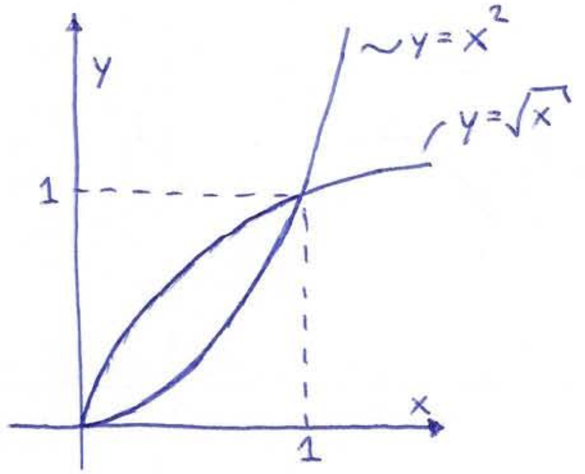


(2 marks each) Integrate with respect to “x” the following functions.

$$\int 3x^2 + 2\sin(x) - 1 \ dx$$

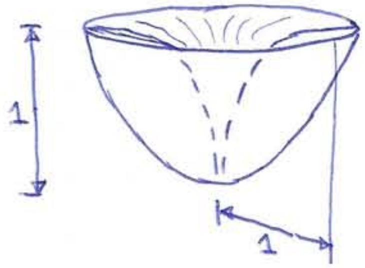
$$\int \frac{2x^2}{3(x^3+1)^{\frac{3}{2}}} + 1 \; dx$$

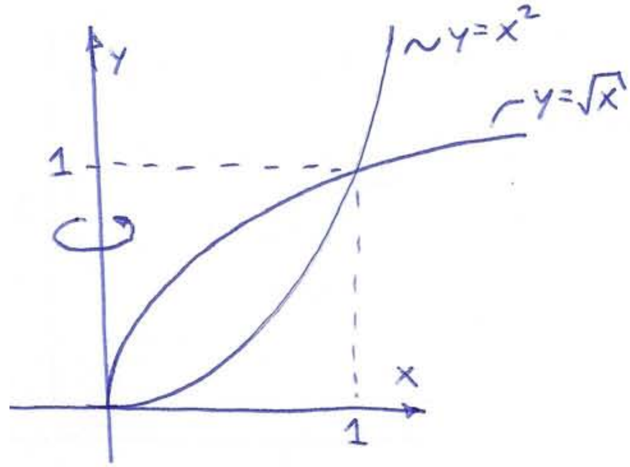
(5 marks) Find the area enclosed by the two functions.



(5 marks) Find the volume between the two functions revolved around the y-axis.
Remember that:

area of a disk = πr^2
area of a shell = $2\pi rh$

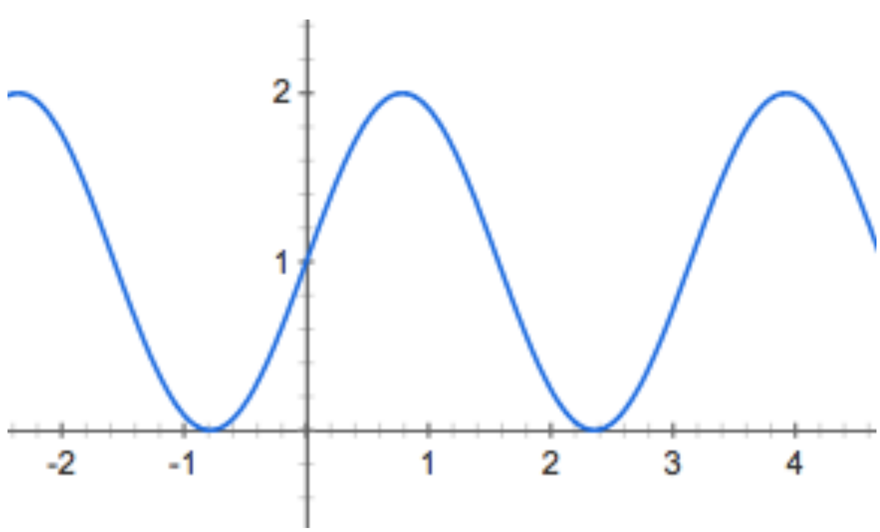




(2 marks) Bonus

Determine the tangent line at x=1 for the following function. Plot the tangent line on the plot.

$$y(x) = \sin(2x) + 1$$



equation of a line:

$$y(x) = ax + b$$

(2 marks) Determine the 101th number in the sequence and the sum from the first number to the 101th number for each of the following series.

23, 29, 35, 41, ...

arithmetic series

$a_1 = 23$

$n = 101$

$k = 29 - 23 = 6$

so

$a_n = a_1 + (n-1)k$

$a_{101} = 23 + (101-1)(6) = 623$

and

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{101} = \frac{101}{2}(23 + 623) = 32623$

$$\boxed{\begin{aligned} a_{101} &= 623 \\ S_{101} &= 32623 \end{aligned}}$$

(2 marks) When you buy a car you could assume that each following year the value of the car will only be 80% of the previous year's value. If you bought a car for 20000$ how much would it be worth in 48 years, assume that the car lasts hat long… it a Tesla!!

20000 , 16000 , 12800 , ...

geometric series

$a_1 = 20000$

$n = 48$

$r = 0.8$

so

$a_n = a_1 r^{n-1}$

$a_{48} = (20000)(0.8)^{48-1}$

$a_{48} = 0.5575$

$$\boxed{\begin{aligned} a_{48} &= 0.56 \ \$ \\ &= 56 \ ¢ \end{aligned}}$$

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 3x^{\frac{2}{3}} + \sin(2x)$$

$$\boxed{\frac{dy}{dx} = 2x^{-\frac{1}{3}} + 2\cos(2x)}$$

$$y(x) = 3x^{\frac{2}{3}} \sin(2x)$$

$$\frac{dy}{dx} = 2x^{-\frac{1}{3}} \sin(2x) + 3x^{\frac{2}{3}}\, 2\cos(2x)$$

$$\boxed{\frac{dy}{dx} = 2x^{-\frac{1}{3}} \sin(2x) + 6x^{\frac{2}{3}} \cos(2x)}$$

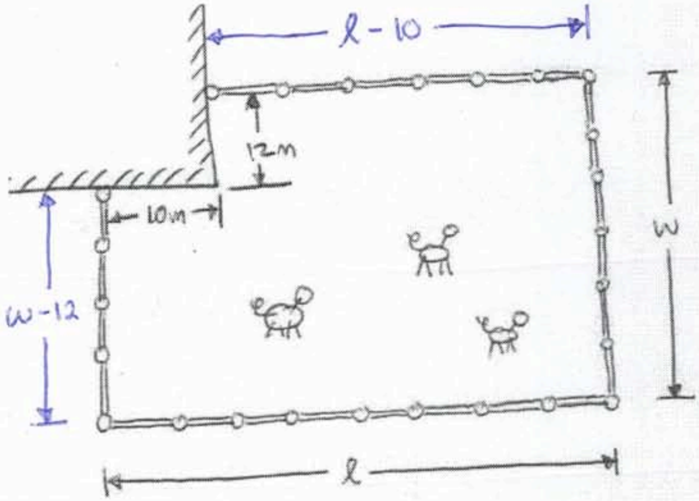
(3 marks) Take the derivative with respect to "x" of the following function

$$y(x) = \left[(x^2+1)^3(x^{\frac{3}{2}}-1)+1\right]^2$$

$$\frac{dy}{dx} = 2\left[(x^2+1)^3(x^{\frac{3}{2}}-1)+1\right]\left[3(x^2+1)^2(2x+0)(x^{3/2}-1) + (x^2+1)^3\left(\frac{3}{2}x^{\frac{1}{2}}-0\right)+0\right]$$

$$\boxed{\frac{dy}{dx} = 2\left[(x^2+1)^3(x^{\frac{3}{2}}-1)+1\right]\left[3(x^2+1)^2(2x)(x^{3/2}-1) + (x^2+1)^3\left(\frac{3}{2}x^{\frac{1}{2}}\right)\right]}$$

(5 marks) You have to build a fence keeping a 10 m x 12 m section that will not be fenced, as shown in the diagram. If you are given 100 m of fencing what dimension would it have to maximize area?



$$100 = (w-12) + l + w + (l-10)$$
$$100 = 2l + 2w - 22$$
$$122 = 2l + 2w \qquad (1)$$

$$A = lw - (10)(12)$$
$$A = lw - 120 \qquad (2)$$

solve for l in (1)

$$2l = 122 - 2w$$
$$l = 61 - w \qquad (1a)$$

sub (1a) into (2)

$$A = [61 - w]w - 120$$
$$A = -w^2 + 61w - 120$$

take derivative and set to zero (max/min)

$$\frac{dA}{dw} = -2w + 61 = 0$$
$$w = \frac{61}{2} = 30.5\text{ m}$$

sub back into (1a)

$$l = 61 - [30.5] = 30.5\text{ m}$$

so

$$\boxed{\begin{array}{c} l = 30.5 \\ w = 30.5 \end{array}}$$

(2 marks each) Integrate with respect to "x" the following functions.

$$\int 3x^2 + 2\sin(x) - 1 \, dx$$

$$\boxed{x^3 - 2\cos(x) - x + C}$$

$$\int \frac{2x^2}{3(x^3+1)^{\frac{3}{2}}} + 1 \, dx$$

$$\int \frac{2x^2}{3(x^3+1)^{3/2}} \, dx \quad + \quad \int 1 \, dx$$

let $u = x^3 + 1$

$$\frac{du}{dx} = 3x^2 \quad \Rightarrow \quad dx = \frac{du}{3x^2}$$

so

$$\int \frac{2\cancel{x^2}}{3u^{3/2}} \, \frac{du}{3\cancel{x^2}} \quad + \quad \int 1 \, dx$$

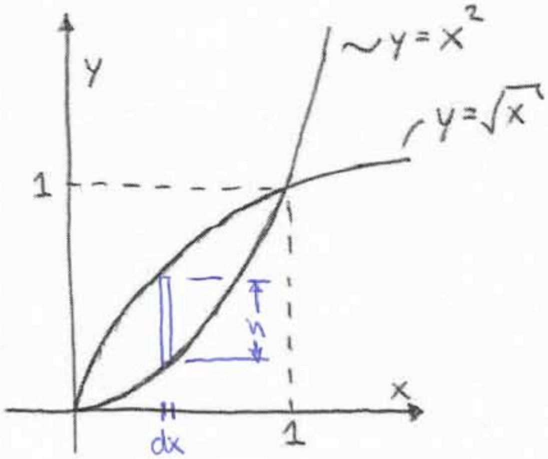
$$\int \frac{2}{9} u^{-3/2} \, du \quad + \quad x \quad + \quad c$$

$$\frac{2}{9}\left(-\frac{2}{1}\right) u^{-\frac{1}{2}} \quad + \quad x \quad + \quad c$$

$$\boxed{-\frac{4}{9}(x^3+1)^{-\frac{1}{2}} + x + c}$$

$$\boxed{-\frac{4}{9(x^3+1)^{\frac{1}{2}}} + x + c}$$

(5 marks) Find the area enclosed by the two functions.



$h = \sqrt{x} - x^2$

using column's

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \int_0^1 x^{\frac{1}{2}} - x^2 \; dx$$

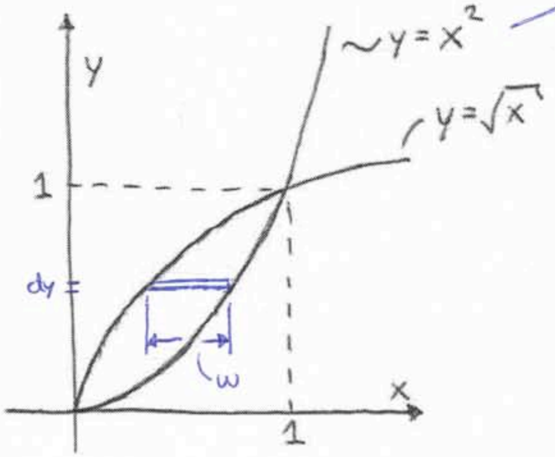
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$$

$$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right] - \left[\cancel{\frac{2}{3}(0)^{\frac{3}{2}}}^{\,0} - \cancel{\frac{1}{3}(0)^3}^{\,0}\right]$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$\boxed{\text{Area} = \frac{1}{3} \text{ units}^2}$$

(5 marks) Find the area enclosed by the two functions.



$\longrightarrow x = \sqrt{y}$

$\longrightarrow x = y^2$

so

$w = \sqrt{y} - y^2$

using row's

$$A = \int_0^1 \left(\sqrt{y} - y^2\right) dy$$

$$= \int_0^1 y^{\frac{1}{2}} - y^2 \, dy$$

$$= \left[\frac{2}{3} y^{\frac{3}{2}} - \frac{1}{3} y^3\right]_0^1$$

$$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right] - \left[\cancel{\frac{2}{3}(0)^{\frac{3}{2}}}^{\,0} - \cancel{\frac{1}{3}(0)^3}^{\,0}\right]$$

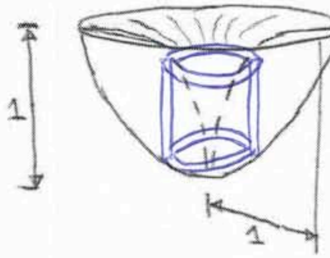
$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

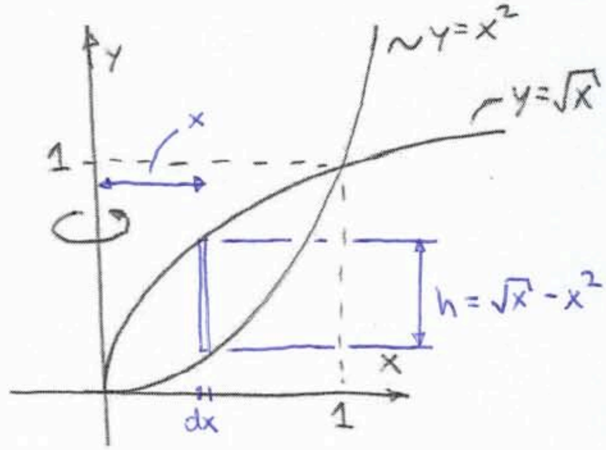
$$\boxed{\text{Area} = \frac{1}{3} \text{ units}^2}$$

(5 marks) Find the volume between the two functions revolved around the y-axis.
Remember that:

area of a disk = πr^2

area of a shell = $2\pi rh$





using shell's

$$dV_{shell} = 2\pi(x)(\sqrt{x}-x^2)\,dx$$

$$V = \int_0^1 2\pi x\left(x^{\frac{1}{2}}-x^2\right)dx$$

$$= \int_0^1 2\pi x^{\frac{3}{2}} - 2\pi x^3 \, dx$$

$$= \left[2\pi \frac{2}{5}x^{\frac{5}{2}} - 2\pi\frac{1}{4}x^4\right]_0^1$$

$$= \left[\frac{4}{5}\pi(1)^{\frac{5}{2}} - \frac{\pi}{2}(1)^4\right] - \left[\cancel{\frac{4}{5}\pi(0)^{\frac{5}{2}}}^{\,0} - \cancel{\frac{\pi}{2}(0)^4}^{\,0}\right]$$

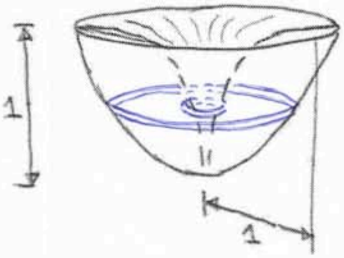
$$= \frac{4\pi}{5} - \frac{\pi}{2} = \frac{3\pi}{10}$$

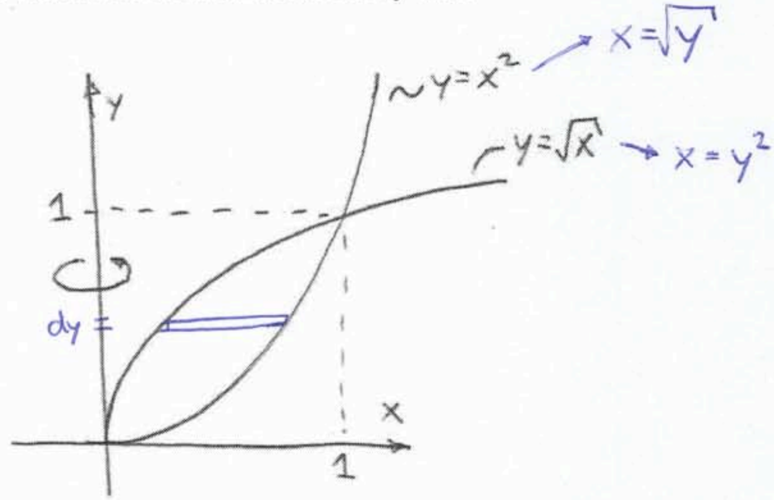
$$\boxed{\text{Volume} = \frac{3\pi}{10}\ \text{units}^3}$$

(5 marks) Find the volume between the two functions revolved around the y-axis.
Remember that:

area of a disk = πr^2

area of a shell = $2\pi rh$





$$dV = \left[\pi\left(\sqrt{y}\right)^2 - \pi\left(y^2\right)^2\right] dy$$

$$dV = \left[\pi y - \pi y^4\right] dy$$

so

$$V = \int_0^1 \pi y - \pi y^4 \, dy$$

$$= \left[\frac{\pi}{2} y^2 - \frac{\pi}{5} y^5\right]_0^1$$

$$= \left[\frac{\pi}{2}(1)^2 - \frac{\pi}{5}(1)^5\right] - \left[\cancel{\frac{\pi}{2}(0)^2}^{\,0} - \cancel{\frac{\pi}{5}(0)^5}^{\,0}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

$$\boxed{\text{Volume} = \frac{3\pi}{10} \text{ units}^3}$$

(2 marks) Bonus

Determine the tangent line at x=1 for the following function. Plot the tangent line on the plot.

$$y(x) = \sin(2x) + 1$$

$$y(1) = \sin(2 \cdot 1) + 1 = 1.91$$

$$\frac{dy}{dx} = 2\cos(2x)$$

$$\frac{dy(1)}{dx} = 2\cos(2 \cdot 1) = -0.832$$

equation of line:

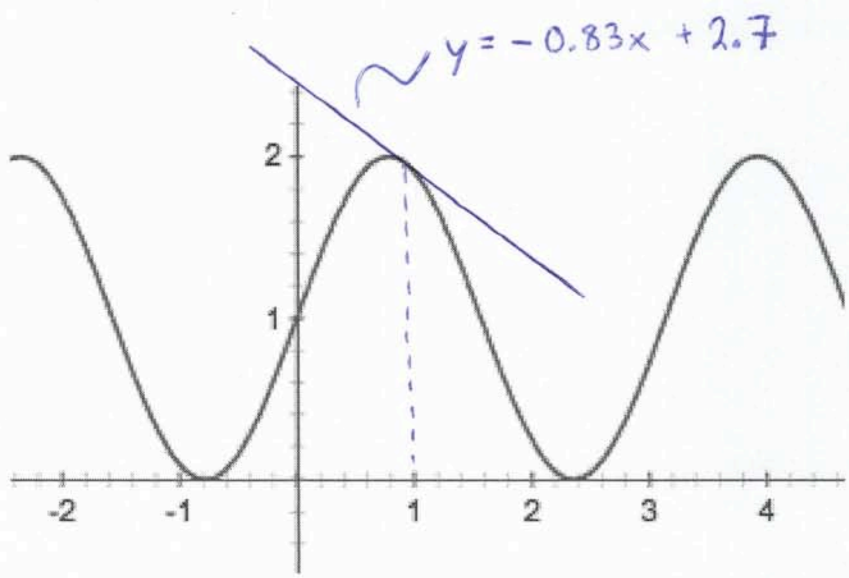
$$y = ax + b$$

so

$$b = y - ax$$
$$= 1.91 - (-0.832)(1) = 2.74$$

so

$$\boxed{y = -0.832x + 2.74}$$



equation of a line:

$$y(x) = ax + b$$