

Instructor: Frank Secretain
 Course: CIVL 11
 Assessment: Final
 Time allowed: 110 minutes
 Devices allowed: Pencil, pen, eraser, calculator
 Notes from instructor: Be neat. Show your work where needed. Box final answers.

Marks allocated: 4 questions worth 25 marks + 1 bonus question worth 2 marks
 Percentage of final grade: 25% of final grade

Body	Figure	Mass Center	Moments of Inertia
Rectangular Parallelepiped		—	$I_{xx} = \frac{1}{12} m(a^2 + l^2)$ $I_{yy} = \frac{1}{12} m(b^2 + l^2)$ $I_{zz} = \frac{1}{12} m(a^2 + b^2)$ $I_{y_1y_1} = \frac{1}{12} mb^2 + \frac{1}{3} ml^2$
Circular Cylinder		—	$I_{xx} = \frac{1}{4} mr^2 + \frac{1}{12} ml^2$ $I_{x_1x_1} = \frac{1}{4} mr^2 + \frac{1}{3} ml^2$ $I_{zz} = \frac{1}{2} mr^2$
Semicylinder		$\bar{x} = \frac{4r}{3\pi}$	$I_{xx} = I_{yy} = \frac{1}{4} mr^2 + \frac{1}{12} ml^2$ $I_{x_1x_1} = I_{y_1y_1} = \frac{1}{4} mr^2 + \frac{1}{3} ml^2$ $I_{zz} = \frac{1}{2} mr^2$
Circular Cylindrical Shell		—	$I_{xx} = \frac{1}{2} mr^2 + \frac{1}{12} ml^2$ $I_{x_1x_1} = \frac{1}{2} mr^2 + \frac{1}{3} ml^2$ $I_{zz} = mr^2$
Half Cylindrical Shell		$\bar{x} = \frac{2r}{\pi}$	$I_{xx} = I_{yy} = \frac{1}{2} mr^2 + \frac{1}{12} ml^2$ $I_{x_1x_1} = I_{y_1y_1} = \frac{1}{2} mr^2 + \frac{1}{3} ml^2$ $I_{zz} = mr^2$
Sphere		—	$I_{zz} = \frac{2}{5} mr^2$
Spherical Shell		—	$I_{zz} = \frac{2}{3} mr^2$
Hemisphere		$\bar{x} = \frac{3r}{8}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$
Hemispherical Shell		$\bar{x} = \frac{r}{2}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} mr^2$
Uniform Slender Rod		—	$I_{yy} = \frac{1}{12} ml^2$ $I_{y_1y_1} = \frac{1}{3} ml^2$

Formula Sheet

Line equation

$$y = ax + b$$

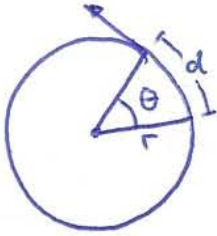
Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Centre of mass

$$\vec{x}_{cg} = \frac{\sum m_i \vec{x}_i}{\sum m_i}$$

Uniform circular motion



$$d = \theta r$$

$$v = \omega r$$

$$a = \alpha r$$

$$a_r = -\omega^2 r = -\frac{v^2}{r}$$

Kinematics

$$\vec{x} = \vec{x}_o + \vec{v}_o t + \frac{1}{2} \vec{a} t^2$$

$$\vec{v}^2 = \vec{v}_o^2 + 2\vec{a}(\vec{x} - \vec{x}_o)$$

$$\vec{v} = \vec{v}_o + \vec{a} t$$

$$\theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$$

$$\omega = \omega_o + \alpha t$$

Dynamics (Newton's 2nd Law)

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a} \quad \vec{F}_g = \frac{Gm_1 m_2}{r^2} = mg \quad \vec{F}_f = \mu N$$

Moments (integral of Newton's 2nd Law with space)

$$\sum \vec{M}_p = \frac{d(I_p \vec{\omega}_p)}{dt} = I_p \vec{\alpha}_p \quad \vec{M} = \vec{r} \times \vec{F} = \vec{F} \perp \vec{r}$$

Conservation of momentum (integral of Newton's 2nd Law with time)

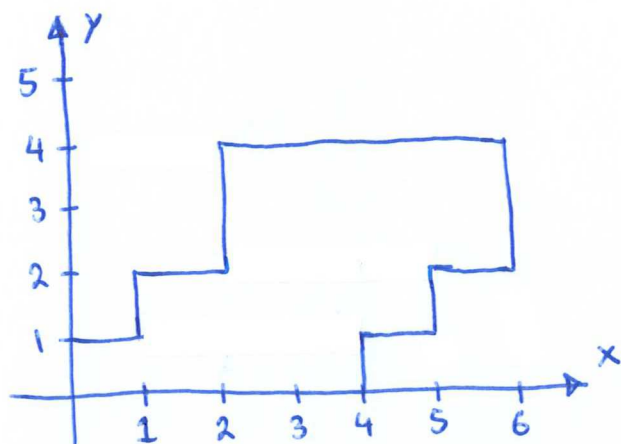
$$\vec{p} = m\vec{v} \quad \vec{p} - \vec{p}_o = \int \vec{F} dt = \vec{F}_{ave.}(t - t_o)$$

1st Law of Thermodynamics

$$E_f - E_o = Q - W \quad E_{PE} = mg(y - y_o) \quad E_{KE} = \frac{1}{2} m v^2$$

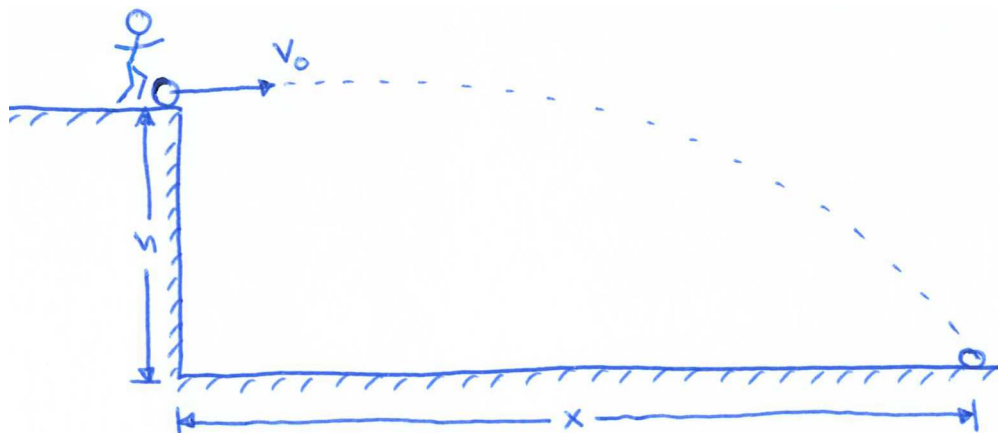
$$W = \vec{F} \cdot (\vec{x} - \vec{x}_o)$$

(3 marks) Determine the centre of mass of the object shown below and draw it on the diagram.



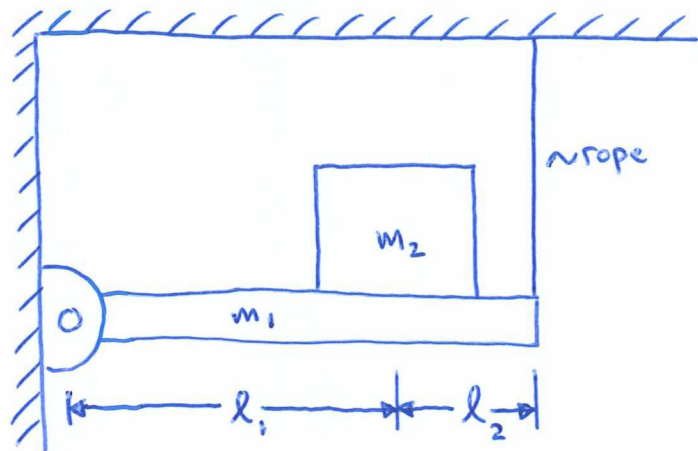
(6 marks) Determine how far the ball goes (x) and the final velocity (velocity magnitude) of the ball just before it hits the ground.

Given: $v_o = 10 \text{ m/s}$
 $h = 5 \text{ m}$



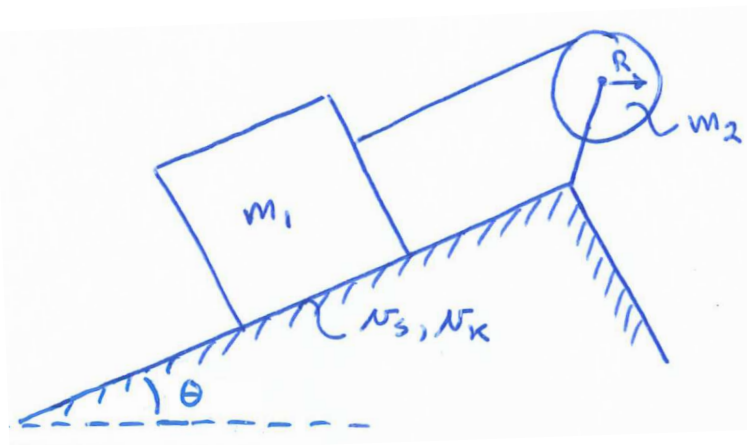
(6 marks) Determine the tension in the rope and the magnitude of the reaction force in the pin.

Given: $m_1 = 10 \text{ kg}$
 $m_2 = 50 \text{ kg}$
 $L_1 = 4 \text{ m}$
 $L_2 = 2 \text{ m}$



(10 marks) Determine if the system moves and if so, what is the magnitude of the acceleration. Assume that the pulley is a solid cylinder with radius R .

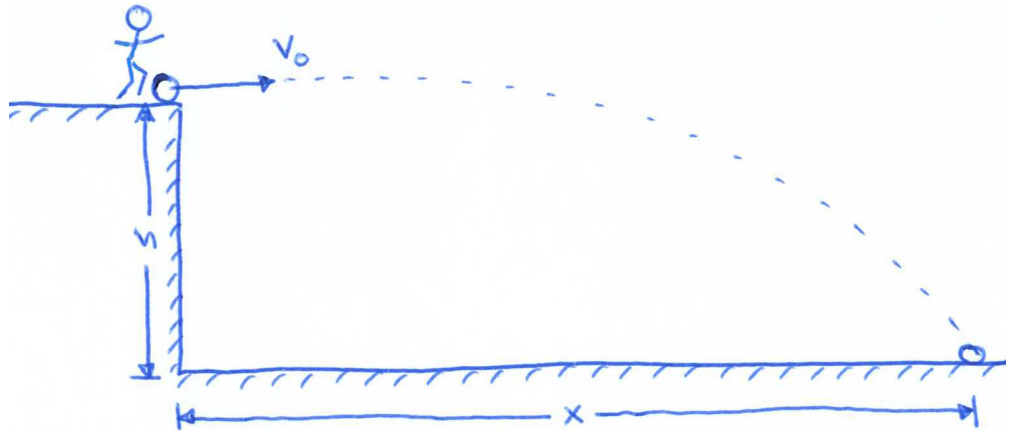
Given: $m_1 = 50 \text{ kg}$
 $m_2 = 10 \text{ kg}$
 $R = 0.5 \text{ m}$
 $\mu_s = 0.2$
 $\mu_k = 0.1$
 $\theta = 25 \text{ degrees}$



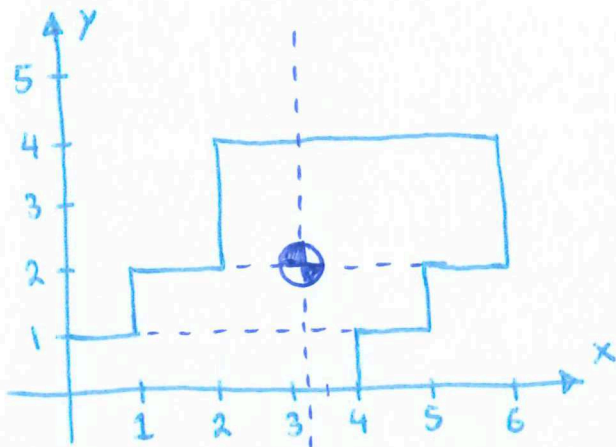
(2 marks) Bonus

Determine the final velocity (velocity magnitude) of the ball just before it hits the ground using the 1st Law of Thermodynamics.

Given: $v_o = 10 \text{ m/s}$
 $h = 5 \text{ m}$



(3 marks) Determine the centre of mass of the object shown below and draw it on the diagram.



$$\vec{x}_{cg} = \frac{\sum m_i \vec{x}_i}{\sum m_i}$$

$$x_{cg} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(4)(2) + (4)(3) + (8)(4)}{4 + 4 + 8} = 3.25$$

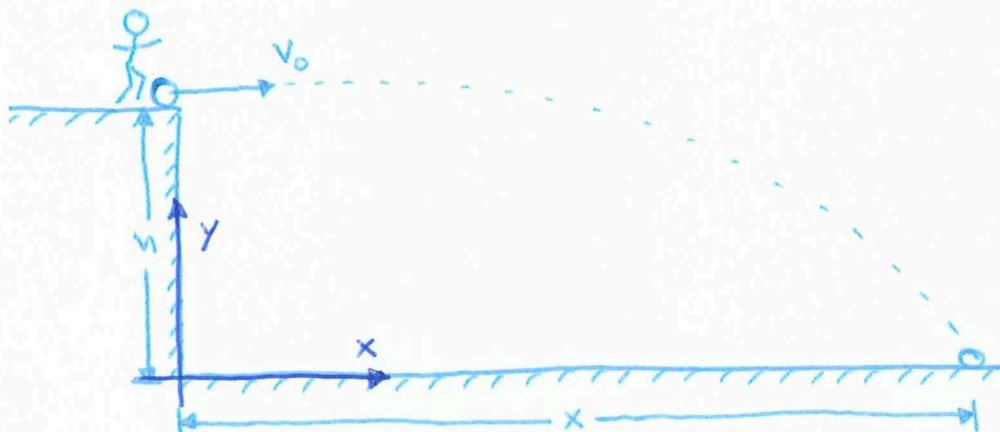
$$y_{cg} = \frac{\sum m_i y_i}{\sum m_i} = \frac{(4)(0.5) + (4)(1.5) + (8)(3)}{4 + 4 + 8} = 2$$

$$x_{cg} = 3.25$$

$$y_{cg} = 2$$

(6 marks) Determine how far the ball goes (x) and the final velocity (velocity magnitude) of the ball just before it hits the ground.

Given: $v_o = 10 \text{ m/s}$
 $h = 5 \text{ m}$



<u>x</u>	<u>y</u>
$x_o = 0$	$y_o = 5$
$x_f =$	$y_f = 0$
$v_{ox} = 10$	$v_{oy} = 0$
$v_{fx} = 10$	$v_{fy} =$
$a_x = 0$	$a_y = -9.81$

$$\vec{v}^2 = \vec{v}_o^2 + 2\vec{a}(\vec{x} - \vec{x}_o)$$

$$v_{fy}^2 = v_{oy}^2 + 2a_y(y_f - y_o)$$

$$v_{fy} = \pm \sqrt{v_{oy}^2 + 2a_y(y_f - y_o)}$$

$$= \pm \sqrt{(0)^2 + 2(-9.81)(0 - 5)} = -9.9 \text{ m/s}$$

$$\vec{v} = \vec{v}_o + \vec{a}t$$

$$v_{fy} = v_{oy} + a_y t$$

$$t = (v_{fy} - v_{oy}) / a_y$$

$$= (-9.9 - 0) / (-9.81) = 1.0 \text{ s}$$

$$\vec{x} = \vec{x}_o + \vec{v}_o t + \frac{1}{2} \vec{a} t^2$$

$$x = x_o + v_{ox} t + \frac{1}{2} a_x t^2$$

$$= 0 + (10)(1.0) + \frac{1}{2} (0)(1.0)^2 = 10.1 \text{ m}$$

$$x = 10 \text{ m}$$

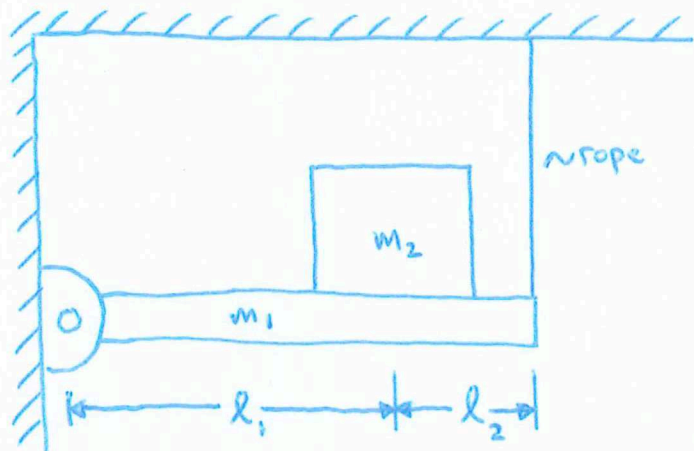
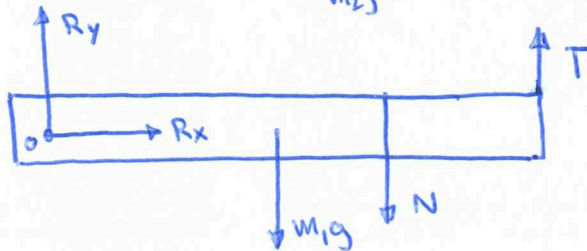
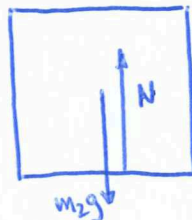
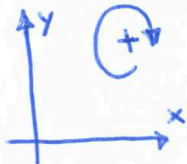
$$|\vec{v}_f| = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(10)^2 + (-9.9)^2} = 14.1$$

$$|\vec{v}_f| = 14 \text{ m/s}$$

(6 marks) Determine the tension in the rope and the magnitude of the reaction force in the pin.

Given: $m_1 = 10 \text{ kg}$
 $m_2 = 50 \text{ kg}$
 $L_1 = 4 \text{ m}$
 $L_2 = 2 \text{ m}$

FBD:



$$\begin{aligned}\sum F_x = m_2 a_x &= 0 \\ \sum F_y = m_2 a_y &= N - m_2 g\end{aligned} \quad (1)$$

$$\sum F_x = m_1 a_x = R_x \quad (2)$$

$$\sum F_y = m_1 a_y = R_y + T - m_1 g - N \quad (3)$$

$$\sum M_o = I_o \alpha_o = (3)(m_1 g) + (4)(N) - (6)(T) \quad (4)$$

solve (1) for N:

$$N = m_2 g \quad (1a)$$

sub (1a) into (4):

$$0 = 3m_1 g + 4[m_2 g] - 6T$$

$$T = \frac{1}{6}(3m_1 g + 4m_2 g)$$

$$= \frac{1}{6}((3)(10)(9.81) + (4)(50)(9.81))$$

$$\boxed{T = 376 \text{ N}} \quad (4a)$$

solve (2) for R_x :

$$R_x = 0$$

sub (1a) & (4a) into (3)

$$0 = R_y + [376.05] - (10)(9.81) - [(50)(9.81)]$$

$$R_y = 212.55$$

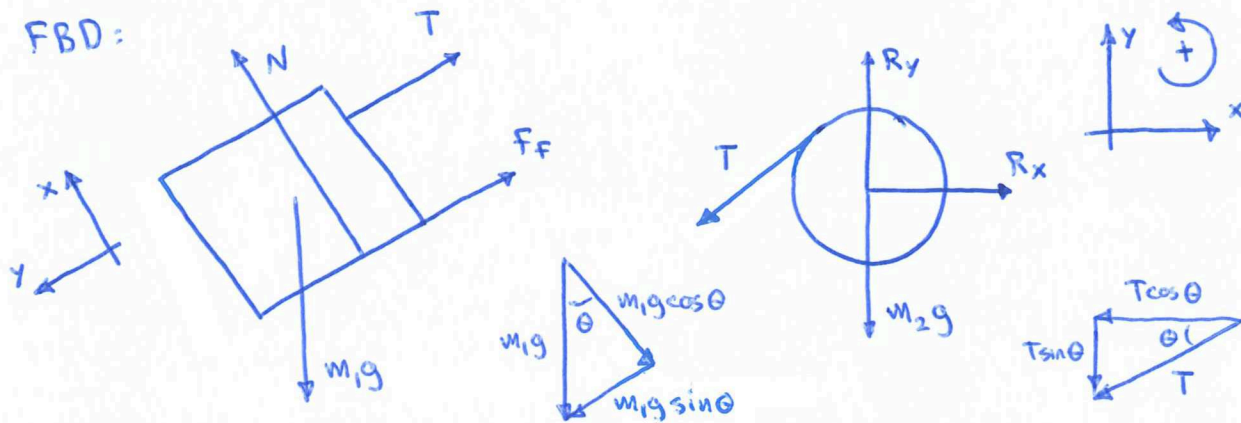
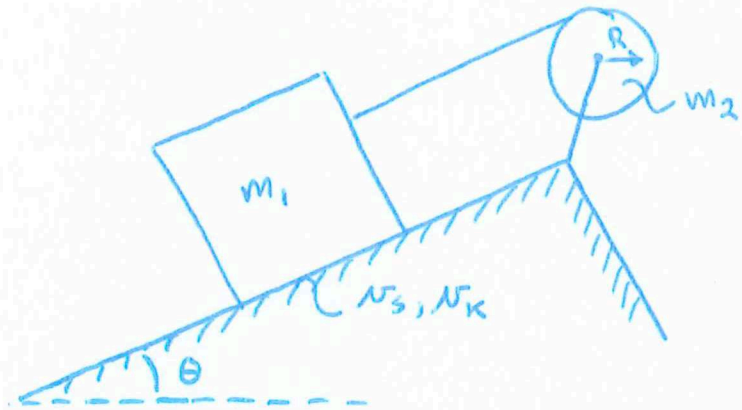
Reaction Force Magnitude:

$$\begin{aligned}|\vec{R}| &= \sqrt{R_x^2 + R_y^2} \\ &= \sqrt{(0)^2 + (212.55)^2}\end{aligned}$$

$$\boxed{|\vec{R}| = 213 \text{ N}}$$

(10 marks) Determine if the system moves and if so, what is the magnitude of the acceleration. Assume that the pulley is a solid cylinder with radius R .

Given: $m_1 = 50 \text{ kg}$
 $m_2 = 10 \text{ kg}$
 $R = 0.5 \text{ m}$
 $\mu_s = 0.2$
 $\mu_k = 0.1$
 $\theta = 25^\circ$



$$\sum F_x = m_1 a_x = m_1 g \sin \theta - T - F_f \quad (1)$$

$$\sum F_y = m_1 a_y = N - m_1 g \cos \theta \quad (2)$$

$$\sum F_x = m_2 a_x = R_x - T \cos \theta \quad (3)$$

$$\sum F_y = m_2 a_y = R_y - T \sin \theta - m_2 g \quad (4)$$

$$\sum M_o = I_o \alpha_o = T R \quad (5)$$

friction: $F_f = \mu N$ (6)

circular motion $a = \alpha R$ (7)

sub (5) & (6) into (1):

$$m_1 a_x = m_1 g \sin \theta - \left[\frac{I_o \alpha_o}{R} \right] - [\mu N] \quad (1a)$$

sub (2) & (7) into (1a)

$$m_1 a_x = m_1 g \sin \theta - \frac{I_o}{R} \left[\frac{a_x}{R} \right] - \mu [m_1 g \cos \theta] \quad (1b)$$

moment of Inertia of pulley $I_o = \frac{1}{2} m_2 R^2$ (8)

sub (8) into (1b)

$$m_1 a_x = m_1 g \sin \theta - \frac{a_x}{R^2} \left[\frac{1}{2} m_2 R^2 \right] - \mu m_1 g \cos \theta$$

$$m_1 a_x = m_1 g \sin \theta - \frac{1}{2} m_2 a_x - \mu m_1 g \cos \theta$$

$$m_1 a_x + \frac{1}{2} m_2 a_x = m_1 g \sin \theta - \mu m_1 g \cos \theta$$

$$a_x \left(m_1 + \frac{1}{2} m_2 \right) = m_1 g (\sin \theta - \mu \cos \theta)$$

$$a_x = m_1 g \frac{\sin \theta - \mu \cos \theta}{m_1 + \frac{m_2}{2}}$$

with $\mu = \mu_s =$ $a_x = 2.15 \text{ m/s}^2$ (it moves).

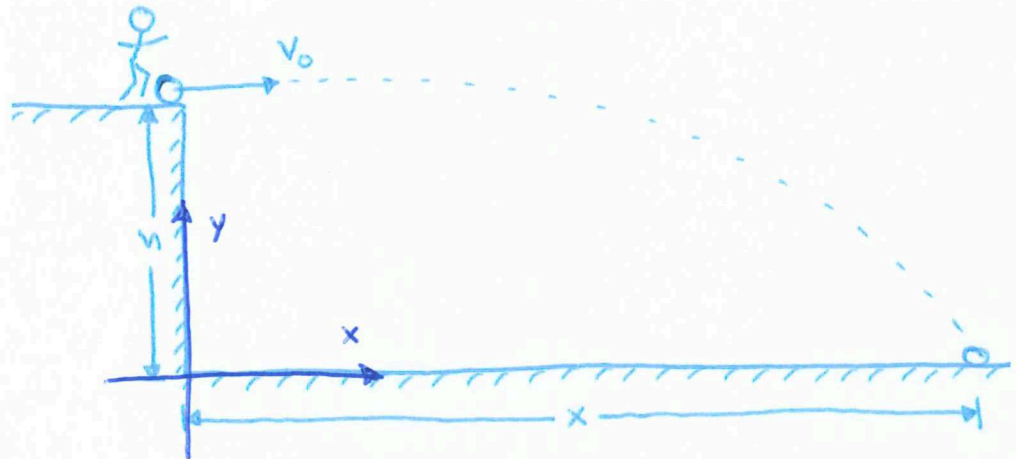
with $\mu = \mu_k =$ $a_x = 2.96 \text{ m/s}^2$

$$a_x = 3.0 \text{ m/s}^2$$

(2 marks) Bonus

Determine the final velocity (velocity magnitude) of the ball just before it hits the ground using the 1st Law of Thermodynamics.

Given: $v_o = 10 \text{ m/s}$
 $h = 5 \text{ m}$



$$E_o = \frac{1}{2} m v_o^2 + m g h_o$$

$$E_i = \frac{1}{2} m v_i^2 + \cancel{m g h_i}$$

1st Law of Thermodynamics:

$$E_i - E_o = \cancel{Q} - \cancel{W}$$

$$\left[\frac{1}{2} m v_i^2 \right] - \left[\frac{1}{2} m v_o^2 + m g h_o \right] = 0$$

$$\frac{1}{2} \cancel{m} v_i^2 = \frac{1}{2} \cancel{m} v_o^2 + \cancel{m} g h$$

$$v_i^2 = v_o^2 + 2 g h$$

$$v_i = \sqrt{v_o^2 + 2 g h}$$

$$= \sqrt{(10)^2 + 2(9.81)(5)}$$

$$v_i = 14 \text{ m/s}$$